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EXISTENCE OF EXTREMAL SOLUTION ABSTRACT MEASURE INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract- In this chapter, an existence result for perturbed abstract measure differential equation are proved by using Leray-Schauder nonlinear alternative, under the caratheodory condition. The existence of external solutions is also proved, under certain monotonicity conditions.

INTRODUCTION

The study of abstract measure differential equations is initiated by Sharma and subsequently developed by Joshi, Shendge and Joshi [and Dhage. Similarly, the study of abstract measure Integro- differential equations is studied by Dhage and Bellale for various aspects of solutions. In such models of differential equations, the ordinary derivative is replaced by the derivative of set functions which thereby gives the generalizations of the ordinary and measure differential equations. The various aspects of the solutions of the abstract measure differential equations have been studied in the literature using the fixed point

techniques such as Schauder’s fixed point principle and Banach contraction mapping principle etc. In such situations one needs to show that the operator under consideration maps a certain set into itself. This is a severe restriction, which motivated us to pursue the study of abstract measure integro-differential equations using the nonlinear alternative of Schauder. The existence of extremal solutions for the abstract measure-integro differential equation in question is also proved under certain monotonicity conditions of the nonlinearity involved in the equations. In the following section we give some preliminaries needed in the sequel.

DEFINITIONS AND NOTATIONS

Let X be a real Banach space with a norm $\| \cdot \|$. Let $x, y \in X$. Then the line segment xy in X is defined by

$$\overline{xy} = \{ z \in X \mid z = x + r(y - x), 0 \leq r \leq 1 \}. \tag{2.1}$$

Let $x_0 \in X$ be a fixed point and $z \in X$. Then for any $x \in \overline{x_0 z}$, we define the sets S_x and \overline{S}_x in X by

$$S_x = \{ rx \mid -\infty < r < 1 \} \tag{2.2}$$

and $\overline{S}_x = \{ rx \mid -\infty < r \leq 1 \}. \tag{2.3}$

Thus we have

$$\overline{xy} = \overline{S}_y - S_x$$

for all $x, y \in X$.

Let $x_1, x_2 \in \overline{x_0 y}$ be arbitrary. We say $x_1 < x_2$ if $S_{x_1} \subset S_{x_2}$ or equivalently $\overline{x_0 x_1} \subset \overline{x_0 x_2}$. In this case we also write $x_2 > x_1$.

Let M denote the σ -algebra of all subsets of X so that (X, M) becomes a measurable space. Let $ca(X, M)$ be the space of all vector measures (signed measures) and define a norm $|\cdot|$ on $ca(X, M)$ by

$$\|p\| = |p|(X) \tag{2.4}$$

Where, $|p|$ is a total variation measure of p and is given by

$$|p|(X) = \sup_{\sigma} \sum_{i=1}^{\infty} |p(E_i)|, \quad \forall E_i \subset X. \quad (2.5)$$

It is known that $ca(X, M)$ is a Banach space with respect to the norm $\|\cdot\|$ defined by (2.2.4). Let μ be a σ -finite measure on X and let $p \in ca(X, M)$. We say p is absolutely continuous with respect to the measure μ if $\mu(E) = 0$ implies $p(E) = 0$ for some $E \in M$. In this case we write $p \ll \mu$.

For a fixed $x_0 \in X$, let M_0 denote the σ -algebra on S_{x_0} . Let $z \in X$ be such that $z > x_0$ and let M_z denote the σ -algebra of all sets containing M_0 and the sets of the form \bar{S}_x for $x \in \overline{x_0 z}$. Finally let $L^1_{\mu}(S_z, \mathbb{R})$ denote the space of all μ integrable real valued functions h on S_z with the norm $\|\cdot\|_{L^1_{\mu}}$ defined by

$$\|h\|_{L^1_{\mu}} = \int_{S_z} |h(x)| d\mu.$$

STATEMENT OF THE PROBLEM

Let μ be a real or σ -finite positive measure on X . Given a $p \in ca(X, M)$ with $p \ll \mu$, consider the abstract measure integro-differential equation (in short AMIDE)

$$\left. \begin{aligned} \frac{dp}{d\mu} &= f(x, p(\bar{S}_x), \int_{\bar{S}_x} k(t, p(\bar{S}_t)) d\mu), \\ &\text{a.e. } [\mu] \text{ on } \overline{x_0 z} \end{aligned} \right\} \quad (3.1)$$

$$p(E) = q(E), \quad E \in M_0,$$

where q is a given known vector measure, $\frac{dp}{d\mu}$ is a Radon-Nikodym derivative of p with respect to μ and $f : S_z \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is such that $x \rightarrow f(x, p(\bar{S}_x), \int_{\bar{S}_x} k(t, p(\bar{S}_t)) d\mu)$ is μ -integrable for each $p \in ca(S_z, M_z)$.

The AMIDE (3.1) is equivalent to the abstract measure integral equation

$$p(E) = \begin{cases} p(E) = \int_E f(x, p(S_x), \int_{\bar{S}_x} k(t, p(\bar{S}_t)) d\mu) d\mu, & E \in M_z, E \subset \overline{x_0 z} \\ q(E), & E \in M_0. \end{cases} \tag{3.2}$$

A solution p of AMIDE (3.1) on $\overline{x_0 z}$ will be denoted by $p(\bar{S}_{x_0}, q)$. we shall prove the main existence of external solution for AMIDE (3.1) under suitable conditions on f . We shall use the following form of the Leray-Schauder's nonlinear alternative.

We consider the following set of assumptions.

(A₁) For any $z > x_0$, the σ -algebra M_z is compact with respect to the topology generated by the Pseudo-metric d defined by $d(E_1, E_2) = |\mu|(E_1 \Delta E_2)$, $E_1, E_2 \in M_z$.

(A₂) q is continuous on M_z with respect to the pseudo-metric d defined in (A₁).

(A₃) The function $f(x, y_1, y_2)$ is L^1_μ - Caratheodory.

(A₄) The function $k : S_z \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous function and

there exists a function $\alpha \in L^1_\mu (J, \mathbb{R}^+)$ such that

$$|k(x, y)| \leq \alpha(x) |y| \quad \text{a.e. } t, s \in J \text{ and for all } y \in \mathbb{R}.$$

(A₅) There exists a function $\phi \in L^1_\mu (S_z, \mathbb{R}^+)$ such that $\phi(x) > 0$

a.e. $[\mu]$, $x \in S_z$ and a \mathcal{D} -functions $\psi : [0, \infty) \rightarrow (0, \infty)$, such that

$$|f(x, y_1, y_2)| \leq \phi(x) \psi(|y_1| + |y_2|) \quad \text{a.e. } [\mu] \text{ on } \overline{x_0 z}$$

for all $y_1, y_2 \in \mathbb{R}$.

and (B₁) The function k satisfies

$$|k(x, y_1) - k(x, z_1)| \leq |y_1 - z_1|, \quad \text{a.e. } [\mu] \text{ on } \overline{x_0 z}$$

for all $y_1, z_1 \in \mathbb{R}$,

(B₂) there exists a function $l \in L^1_\mu(S_z, M_z)$ such that

$$|f(x, y_1, y_2) - f(x, z_1, z_2)| \leq l(x) [|y_1 - z_1| + |y_2 - z_2|]$$

a.e. $[\mu]$ on $\overline{x_0 z}$, for all $y_1, y_2, z_1, z_2 \in \mathbb{R}$.

EXISTENCE OF EXTREMAL SOLUTIONS

In this section we prove the existence of extremal solutions to AMIDE (3.1) under certain monotonicity conditions on the nonlinearity f involved in (3.1). The existence is obtained when f is Caratheodory as well as discontinuous on the domain of its definition.

We define an order relation \leq in $ca(S_z, M_z)$ by the cone K in $ca(S_z, M_z)$ defined by

$$K = \{ p \in ca(S_z, M_z) \mid p(E) \geq 0 \quad \forall \quad E \in M_z \}. \quad (4.1)$$

Thus, for any $p_1, p_2 \in ca(S_z, M_z)$, we have

$$p_1 \leq p_2 \quad \Leftrightarrow \quad p_2 - p_1 \in K$$

or, equivalently,

$$p_1 \leq p_2 \quad \Leftrightarrow \quad p_1(E) \leq p_2(E)$$

for all $E \in M_z$.

We consider the following assumptions.

(C₁) The function $f(x, y_1, y_2)$ is non decreasing in y_1, y_2 almost everywhere $[\mu]$ on $x_0 z$.

(C₂) AMIDE (2.3.1) has a lower solution u and an upper solution v on M_z with $u \leq v$.

When the function $f(x, y)$ is not $L^1\mu$ -Caratheodory, i.e. $f(x, y)$ is discontinuous in both the state as well as phase variables x and y , then the existence of the extremal solutions of the AMIDE (3.1) can be proved by using the following form of the fixed point theorem of Heikkila and Lakshmikantham.

Theorem 1 : Suppose that the assumptions $(A_1) - (A_3)$ and $(C_1) - (C_2)$ hold. Then AMIDE (3.1) has a minimal and a maximal solution in $[u, v]$.

Proof : Let $X = ca(S_z, M_z)$ and consider the order interval $[u, v]$ in X . Clearly the order interval $[u, v]$ is well defined in view of assumption (C_2) . Define an operator T on $[u, v]$. We shall show that T satisfies all the conditions of Theorem.

Define a function $h : \overline{x_0 z} \rightarrow \mathbb{R}$ by

$$h(x) = \left| f \left(x, u(\overline{S}_x), \int_{\overline{S}_x} k(t, u(\overline{S}_t)) d\mu \right) \right| + \left| f \left(x, v(\overline{S}_x), \int_{\overline{S}_x} k(t, v(\overline{S}_t)) d\mu \right) \right|.$$

Then h is μ -integrable and satisfies

$$\left| f \left(x, p(\overline{S}_x), \int_{\overline{S}_x} k(t, p(\overline{S}_t)) d\mu \right) \right| \leq h(x)$$

a.e. $[\mu]$ on $\overline{x_0 z}$, for all $p \in [u, v]$.

Let $\{p_n\}$ be a monotone increasing sequence in $[u, v]$. We shall show that Tp_n converges to a point in $[u, v]$. Now by the definition of T we obtain

$$Tp_n(E) = \begin{cases} \int_E f\left(x, p_n(\bar{S}_x), \int_{\bar{S}_x} k(t, p_n(\bar{S}_t))\right) d\mu, & \text{if } E \in M_z, E \subset \overline{x_0z} \\ q(E), & \text{if } E \in M_0. \end{cases} \quad (4.2)$$

To finish, it is enough to prove that $\{Tp_n\}$ is a uniformly bounded and equi-continuous sequence in $[u, v]$. Obviously $\{Tp_n\} \subset [u, v]$ since $T : [u, v] \rightarrow [u, v]$. As T is nondecreasing $\{Tp_n\}$ is also a nondecreasing sequence in $[u, v]$.

First we show that $\{Tp_n\}$ is uniformly bounded. Let $E \in M_z$. Then there exist sets $F \in M_0$ and $E \subset \overline{x_0z}$ such that $E = F \cup G$ and $F \cap G = \phi$. From (4.2) it follows that

$$\begin{aligned} |Tp_n(E)| &\leq |q(F)| + \int_E f\left(x, p_n(\bar{S}_x), \int_{\bar{S}_x} k(t, p_n(\bar{S}_t))\right) d\mu \\ &\leq |q|(F) + \int_E f\left(x, p_n(\bar{S}_x), \int_{\bar{S}_x} k(t, p_n(\bar{S}_t))\right) d\mu \\ &\leq \|q\| + \int_G h(x) d\mu \\ &\leq \|q\| + \int_{x_0z} h(x) d\mu \\ &\leq \|q\| + \|h\|_{L^1_\mu} \end{aligned}$$

for all $E \in M_z$. This further implies that

$$\|Tp_n\| \leq \|q\| + \|h\|_{L^1_\mu}$$

for each $n \in N$ and so, $\{Tp_n\}$ is a uniformly bounded sequence in $[u, v]$.

Next we show that $\{Tp_n\}$ is a equi-continuous sequence of function in X . Let $E_1, E_2 \in M_z$. Then there exist sets $F_1, F_2 \in M_0$ and $G_1, G_2 \in M_z, G_1 \subset \overline{x_0z}, G_2 \subset \overline{x_0z}$ such that $F_i \cap G_i = \phi$ for $i = 1, 2$.

Now

$$Tp_n(E_1) = q(F_1) + \int_{G_1} f\left(x, p_n(\bar{S}_x), \int_{\bar{S}_x} k(t, p_n(\bar{S}_t)) d\mu\right) d\mu$$

and
$$Tp_n(E_2) = q(F_2) + \int_{G_2} f\left(x, p_n(\bar{S}_x), \int_{\bar{S}_x} k(t, p_n(\bar{S}_t)) d\mu\right) d\mu$$

Therefore we have

$$|Tp_n(E_1) - Tp_n(E_2)| \leq |q(F_1) - q(F_2)| +$$

$$\left| \int_{G_1} f\left(x, p_n(\bar{S}_x), \int_{\bar{S}_x} k(t, p_n(\bar{S}_t)) d\mu\right) d\mu - \int_{G_2} f\left(x, p_n(\bar{S}_x), \int_{\bar{S}_x} k(t, p_n(\bar{S}_t)) d\mu\right) d\mu \right| \tag{4.3}$$

$$|Tp_n(E_1) - Tp_n(E_2)| \leq |q(F_1) - q(F_2)| +$$

$$\begin{aligned} & \left| \int_{G_1 - G_2} f\left(x, p_n(\bar{S}_x), \int_{\bar{S}_x} k(t, p_n(\bar{S}_t)) d\mu\right) d\mu - \int_{G_2 - G_1} f\left(x, p_n(\bar{S}_x), \int_{\bar{S}_x} k(t, p_n(\bar{S}_t)) d\mu\right) d\mu \right| \\ & \leq |q(F_1) - q(F_2)| + \int_{G_1 \Delta G_2} \left| f\left(x, p_n(\bar{S}_x), \int_{\bar{S}_x} k(t, p_n(\bar{S}_t)) d\mu\right) \right| d\mu \\ & \leq |q(F_1) - q(F_2)| + \int_{G_1 \Delta G_2} h(x) d\mu. \end{aligned} \tag{4.4}$$

Now assume that $d(E_1, E_2) = |\mu|(E_1 \Delta E_2) \rightarrow 0$. Then we have $E_1 \rightarrow E_2$ and consequently $d(F_1, F_2) \rightarrow 0$ and $|\mu|(G_1 \Delta G_2) \rightarrow 0$. Since the function q is continuous on a compact space M_0 , it is uniformly continuous. As a results $|q(F_1) - q(F_2)| \rightarrow 0$.

Now from (4.4) it follows that

$$|Tp_n(E_1) - Tp_n(E_2)| \rightarrow 0 \text{ as } E_1 \rightarrow E_2.$$

This shows that $\{Tp_n : n \in \mathbb{N}\}$ is an equi-continuous set in $[u, v]$. Hence an application of Arzella – Ascoli theorem yields that the set

$\{ Tp_n : n \in \mathbb{N} \}$ is totally bounded . As a result $\{ Tp_n : n \in \mathbb{N} \}$ has a convergent subsequence converging to a point in $[u, v]$. Since $\{ Tp_n : n \in \mathbb{N} \}$ is monotone increasing , the whole sequence converges in $[u, v]$. Thus the mapping T satisfies all the conditions of Theorem 2.6.1 and hence an application of it yields that T has a least and a greatest fixed point in $[u, v]$. As a result the AMIDE (3.1) has a minimal and a maximal solution in $[u, v]$. This completes the proof .

Notice that hypothesis (C_1) in Theorem may also be replace with the following assumption

- (D₁) There exists a μ -integrable function $h : \overline{x_0z} \rightarrow \mathbb{R}$ such that
- $$|f(x, y_1, y_2)| \leq h(x) \quad \text{a.e. } [\mu] \text{ on } \overline{x_0z}$$
- for all $y \in \mathbb{R}$.

Theorem 2 : Suppose that hypothesis $(A_1) - (A_3)$, (C_1) and (D_3) hold. Then AMIDE admits a minimal and a maximal solution on M_z .

Proof : To conclude, we just show that assumption (B_2) is satisfied. Define a vector measure u and v on $\overline{x_0z}$ by

$$u(E) = \begin{cases} - \int_E h(x) d\mu, & \text{if } E \in M_z, E \subset \overline{x_0z} \\ q(E), & \text{if } E \in M_0, \end{cases}$$

and

$$v(E) = \begin{cases} \int_E h(x) d\mu, & \text{if } E \in M_z, E \subset \overline{x_0z} , \\ q(E), & \text{if } E \in M_0 . \end{cases}$$

We shall show that the vector measures u and v serve as the lower and upper solution of AMIDE (3.1) respectively. From (C_3) it follows that

$$- h(x) \leq f(x, y_1, y_2) \leq h(x) \text{ a.e. } |\mu| \text{ on } \overline{x_0Z},$$

for all $y_1, y_2 \in \mathbb{R}$

In particular, we have

$$- h(x) = f\left(x, p(\overline{S_x}), \int_E f(x, p(\overline{S_x})) d\mu\right) \leq h(x) \text{ a.e. } |\mu| \text{ on } \overline{x_0Z}.$$

Therefore,

$$\begin{aligned} u(E) &= -\int_E h(x) d\mu \\ &\leq \int_E f\left(x, p(\overline{S_x}), \int_E f(x, p(\overline{S_x})) d\mu\right) d\mu \\ &\leq \int_E h(x) d\mu \\ &= v(E) \end{aligned}$$

for all $E \in M_z, E \subset \overline{x_0Z}$

Thus, the vector measures u and v are respectively the lower and upper solutions for AMIDE (3.1) on $\overline{x_0Z}$ with $u \leq v$. Hence (C_2) is satisfied. Now the desired conclusion follows by an application of Theorem . This completes the proof.

Again note that the conclusion of Theorem also remains true if we replace hypothesis (C_2) by

(D_4) There exists a μ -integrable continuous nondecreasing function $g : S_z \times \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$|f(x, y_1, y_2)| \leq g(x, |y_1|, |y_2|) \text{ a.e. } |\mu| \text{ on } \overline{x_0Z}.$$

for all $y_1, y_2 \in \mathbb{R}$ and that AMIDE

$$\frac{dp}{d\mu} = g\left(x, p(\overline{S_x}), \int_{\overline{S_x}} k(t, p(\overline{S_t}))\right),$$

$$p(E) = |q(E)|$$

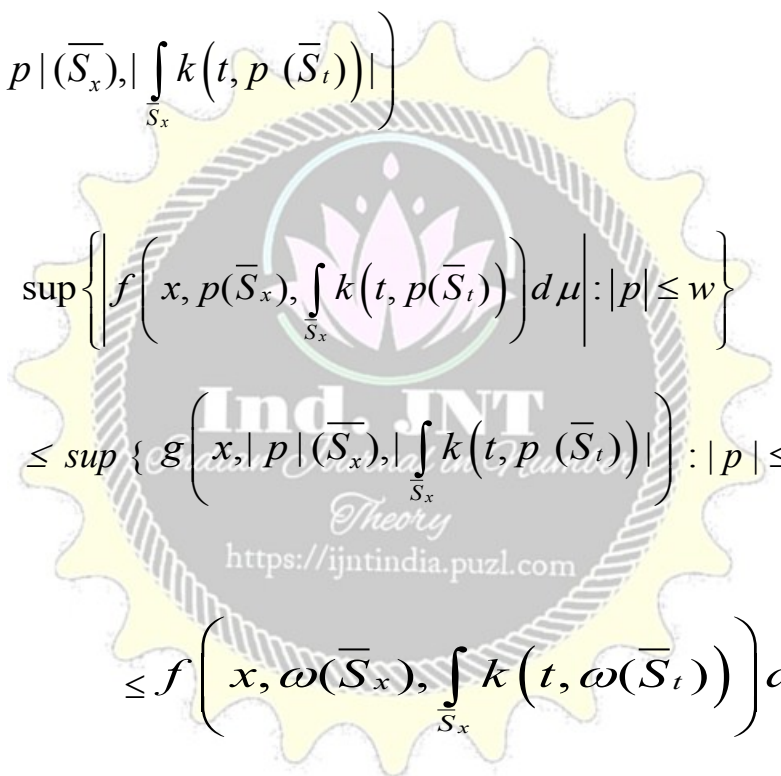
has an upper solution w on $\overline{x_0z}$.

Theorem 3: Suppose that assumptions $(A_1) - (A_3)$, (C_1) and (D_4) hold. Then AMIDE (3.1) has a minimal and a maximal solution on $\overline{x_0z}$.

Proof : Since (C_1) holds, we have

$$\left| f \left(x, p(\overline{S}_x), \int_{\overline{S}_x} k(t, p(\overline{S}_t)) \right) d\mu \right| \leq g \left(x, |p|(\overline{S}_x), \left| \int_{\overline{S}_x} k(t, p(\overline{S}_t)) \right| \right)$$

Therefore,



$$\begin{aligned} & \sup \left\{ \left| f \left(x, p(\overline{S}_x), \int_{\overline{S}_x} k(t, p(\overline{S}_t)) \right) d\mu \right| : |p| \leq w \right\} \\ & \leq \sup \left\{ g \left(x, |p|(\overline{S}_x), \left| \int_{\overline{S}_x} k(t, p(\overline{S}_t)) \right| \right) : |p| \leq w \right\} \\ & \leq f \left(x, \omega(\overline{S}_x), \int_{\overline{S}_x} k(t, \omega(\overline{S}_t)) \right) d\mu \\ & \leq \frac{d\omega}{d\mu} \quad \text{a.e. } |\mu| \text{ on } \overline{x_0z}. \end{aligned}$$

Therefore we have

$$-\frac{d\omega}{d\mu} \leq f \left(x, \omega(\overline{S}_x), \int_{\overline{S}_x} k(t, \omega(\overline{S}_t)) \right) d\mu \leq \frac{d\omega}{d\mu}$$

a.e. $|\mu|$ on $\overline{x_0z}$,

and

$$-w(E) \leq q(E) \leq w(E), \quad \text{if } E \in M_0.$$

This shows that the vector measures $-w$ and w serve respectively as the lower and upper solutions of the AMIDE (3.1) on $\overline{x_0z}$. Thus hypothesis (C_2) is satisfied. Now the desired conclusion follows by an application of Theorem 1. This completes the proof.

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