Ind. JNT
(Indian Journal in Number theory)
Received: *21.04.2015* ***Year:*** *2017,* ***Pages:*** *72-82*
Published: *30.11.2017* *Original Article***

EXISTENCE THEOREM FOR NONLINEAR FUNCTIONAL TWO POINT BOUNDARY VALUE IN BANACH ALGEBRAS

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Abstract- In this paper, a nonlinear functional differential equation is studied in in two point boundary value problem in Banach algebra. The existence of solution functional differentia equation is proved by the under certain general Lipschitz and Caratheodory conditions. Also we have proved the existence of external solution.

Keywords: Banach algebra, Lipschitz and Caratheodory conditions, Nonlinear Functional Differential Equation

INTRODUCTION:

In Given a closed and bounded interval $J = [a, b]$, $a < b$, of real numbers **R**, consider the nonlinear two point functional boundary value problem (in short FBVP) of second order neutral differential equation

$$\left.\begin{array}{l} -\left[\dfrac{X(t)}{f(t, x(\mu(t)))}\right]'' = g(t, x(\sigma(t), x'(\eta(t)) \quad \text{a.e. } t \in J, \\ x(a) = x'(b) = 0, \end{array}\right\} \qquad (1.1)$$

where $f: J \times \mathbf{R} \rightarrow \mathbf{R} - \{0\}$, $g: J \times \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ and $\mu, \sigma, \eta : J \rightarrow J$.

By a solution of the above FBVP we mean a function $x \in AC^1(J, \mathbf{R})$ which satisfies the differential equation and the boundary condition of (1.1), where $AC^1(J, \mathbf{R})$ is the space of all continuous real-valued functions on $J=[a, b]$, whose first derivative is absolutely continuous on J. Note that the second derivative of the solution $x(t)$ exists for almost all $t \in J$. The main idea is to write (1.1) in equivalent operator equation $x=Ax\ Tx$ and to prove that it has a solution in $AC^1(J, \mathbf{R})$.

The FBVP (1.1) has been studied in literature before but results of this paper are new in the theory of differential equations in Banach algebras. The special cases of FBVP (1.1) have already been discussed in the literature by several authors for various aspects of the solution. For example, if $f(t, x) = 1$ whenever $(t, x) \in J \times \mathbf{R}$ and $\sigma(t) = \eta(t) = t$ for all $t \in J$, then FBVP (1.1) reduces to

$$\left.\begin{array}{l} -x''(t) = g(t, x(t), x'(t)) \text{ for a.e. } t \in J, \\ x(a) = x'(b) = 0. \end{array}\right\} \quad (1.2)$$

There is a abundance literature on BVP (1.2), see Baily, Bernfield and Lakshmikantham. The importance of FBVP (1.1) in applications is yet to be investigated. However, it is new to literature on the theory of nonlinear two point boundary value problems. This is the main motivation to study FBVP (1.1) in the present paper.

PRELIMINARIES:

Let X be Banach space with a norm $\|.\|$. A mapping $T: X \rightarrow X$ is called *Lipschitz operator* with a *Lipschitz constant* α if

$$\| Tx - Ty \| \leq \alpha \| x-y \| \quad \text{for all } x, y \in X. \quad (2.1)$$

Further if $0 < \alpha < 1$, T is called a *contraction* with a *contraction constant* α.

Operator T: $\Omega \rightarrow X$, $\Omega \subseteq X$, is called *compact* if it is continuous and $T(\Omega)$ is relative compact subset of X. If $T(S)$ is relative compact subset of X whenever $S \subseteq \Omega$ is bounded, then the operator T is *totally bounded*. The operator T is *completely continuous* if it is continuous and totally bounded. Note that every compact operator is completely continuous, but the converse is not true. However the previous notations are equivalent on a bounded subset of X.

Our results are based on a nonlinear alternative of Leray-Schauder type theorem which is an application of Theorem 2.3 in [3].

Theorem 2.1 : *Let K be a convex subset of a normed linear space E, U an open subset of K with $0 \in U$, and $N : U \rightarrow K$ a continuous and compact map. Then either*

(1) *N has a fixed point in U; or,*

(2) *there is an element u of the boundary ∂U and a real number*

$\lambda \in (0, 1)$ *such that* $u = \lambda N u$.

We prove first a theorem concerning the existence of a solution of the operator equation $x = Ax\, Tx$ in Banach algebra X.

Theorem 2.2 : *Let* X *be a Banach algebra and* $B(0, r)$, $0 < r$, *a closed ball centered at the origin. Assume that* $A : X \to X$ *and* $T : B(0, r) \to X$ *are two operators such that*

(a) *is completely continuous and* $M = \sup \{ ||Tx|| \mid x \in B(0, r) \}$ *and*

(b) A *is Lipschitz operator in with a Lipschitz constant* α *such that* $\alpha M < 1$.

Then either

(i) *the operator equation* $x = Ax\, Tx$ *has a solution in* $B(0,r)$; *or,*

(ii) *there is an element* u *of the boundary* $\partial B(0, r)$ *and real number* $\lambda \in (0, 1)$ *such that* $u = \lambda A\, ((1/\lambda)\, u\,) Tu$.

Proposition 2.1 : *Let* X *be a Banach algebra,* $B(0, r_1)$ *and* $B(0, r_2)$, $0 < r_1, r_2$, *closed balls centered at the origin. Assume that* $A : B(0, r_1) \to X$ *and* $T : B(0, r_2) \to X$ *are two operators such that*

(a) T *is completely continuous and* $M = \sup \{||Tx|| \mid x \in B(0, r_2)\}$, *and*

(b) $A[B(0, r_1)] \subseteq B(0, r_1/M)$ *and* A *is Lipschitz operator in* $B(0, r_1)$ *a Lipschitz constant* α, $\alpha M < 1$. *If* $r_1 \leq r_2$ *then the operator equation* $x = Ax\, Tx$ *has a solution in* $B(0, r_1)$. *Otherwise either*

(i) *the operator equation* $x = Ax\, Tx$ *has a solution in* $B(0, r_2)$; *or,*

(ii) *there is an element* u *of the boundary* $\partial B(0, r_2)$ *and real number* $\lambda \in (0, 1)$ *such that* $u = \lambda A((1/\lambda u)) Tu$.

Lemma 2.1 : $AC^1(J, \mathbf{R})$ is a Banach algebra with respect to the multiplication $(xy)(t) = x(t)\, y(t)$, $t \in J$. We denote by $L^1(J, \mathbf{R})$ the space of all Lebesgue integrable functions on J with the norm

$$||x||_{L1} = \int_a^b |\, x(t)\, |\, dt\,. \qquad (2.3)$$

Definition 2.1 : A function $g : J \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is said to be Caratheodory function if

(1) $t \to g(t, x, y)$ is measurable for all $x, y \in \mathbf{R}$,

(2) $(x, y) \to g(t, x, y)$ is continuous for almost all $t \in J$.

EXISTENCE RESULT:

We consider the following set of hypothesis imposed on functions $f : J \times \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R} \setminus \{0\}$, $g : J \times \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ and $\mu, \sigma, \eta : J \to J$:

(A_0) The function $g(t, x, y)$ is a Caratheodory function.

(A_1) There exists a function $\psi \in L^1(J, \boldsymbol{R}_+)$ and an increasing function $\phi : \boldsymbol{R}_+ \to \boldsymbol{R}_+$ such that

$$|g(t, x, y)| \leq \psi(t)\, \phi(\max\{|x|, |y|\}) \tag{3.1}$$

for a.e. $t \in J$ whenever $x, y \in \boldsymbol{R}$.

(A_2) The function $\mu : J \to J$ has absolutely continuous first derivative and

$$m_1 = \sup_{t \in J} |\mu'(t)|, \tag{3.2}$$

t∈Jand functions σ and $\eta : J \to J$ are continuous.

(A_3) There is a real number $R > 0$ and bounded functions p, p_1 and $p_2 : J \to \boldsymbol{R}_+$ such that

$|f(t, x) - f(t, y)| \leq p(t)\, |x - y|$,

$|f_1(t, x) - f_1(t, y)| \leq p_1(t)\, |x - y|$ and

$|f_2(t, x) - f_2(t, y)| \leq p_2(t)\, |x - y|$ for all $t \in J$ whenever $|x|, |y| \leq \boldsymbol{R}$,

and the partial derivatives $f_j(t, x)$, $j = 1, 2$, are continuous in t for each x, $|x| \leq \boldsymbol{R}$

The FBVP (1.1) can be written in equivalent functional integral equation.

The FBVP (1.1) is equivalent to the functional integral equation (in short FIE)

$$x(t) = f(t, x(\mu(t)))\int G(t, s)\, g(s, x(\sigma(s)), x'(\eta(s)))\, ds \tag{3.4}$$

for all $t \in J$, where $G : J \times J \to \boldsymbol{R}$ is the Green's function

$$G(t, s) = \begin{cases} s - a, & a \leq s \leq t \leq b\,; \\ t - a, & a \leq t \leq s \leq b\,. \end{cases} \tag{3.5}$$

It is easy to see that $G(t, s)$ is continuous in $J \times J$ and $G_t(t, s)$ is continuous in $(a, b) \times (a, b) \setminus \{(t, t) \mid t \in J\}$ and they satisfy the inequalities

$|G(t, s)| = G(t, s) \leq b - a$, and

$$|G_1(t, s)| = G_1(t, s) = \begin{cases} 0, & a < s < t < b, \\ 1, & a < t < s < b \end{cases} \leq 1. \tag{3.6}$$

Theorem 3.1 : *Assume that there exist real numbers $r > 0$ and $R > 0$ such that the hypotheses (A_0) - (A_3) hold and*

$$M(r)\max\{M_0, \max\{M_0, m_1c_2\} + m_1M_2R\} < 1, \tag{3.7}$$

where $M(r) = \max\{1, b-a\}\, ||\psi||_{L^1}\, \phi(r)$, $M_0 = \sup_{t \in J} p(t)$, $M_1 = \sup_{t \in J} p_1(t)$,

$M_2 = \sup_{t \in J} p_2(t)$, $c_2 = \sup_{t \in J} \{ |f_2(t, 0)| \mid t \in J \}$,

$$\max\{M_0R + c_0,\ m_1M_2R^2 + (M_1 + m_1c_2)R + c_1\} \leq R/M. \tag{3.8}$$

Where $c_0 = sup\{\ |f(t, 0)|\ |\ t \in J\}$ *and* $c_1 = sup\{\ |f_1(t, 0)|\ |\ t \in J\}$, *and*

$$r > \frac{b-a}{\max\{1, b-a\}} M(r) \max \left\{ \begin{matrix} M_0 r + c_0, \left[\dfrac{M_0}{b-a} + M_1 + m_1 (M_2 R + c_2) r \right] \\ + \dfrac{c_0}{b-a} + c_1 \end{matrix} \right\} \tag{3.9}$$

Then FBVP (*1.1*) *has a solution* $x \in AC^1(J, \mathbf{R})$ *with* $||x||_{AC^1} \leq r$.

Proof : Let $x = AC^1(J, \mathbf{R})$ and consider a closed balls $B(0, r)$, $r > 0$, and $B(0, R)$, $R > 0$, in X where r satisfies the inequalities (3.7) and (3.8).

We show that the first derivative $f_1(t, x(\mu(t))) + \mu'(t)\, x'(\mu(t))\, f_2(t, x(\mu(t)))$ of $f(t, x(\mu(t)))$ is absolutely continuous for each $x \in B(0, R)$ so that $f(t, x(\mu(t))) \in AC^1(J, \mathbf{R})$ whenever $x \in B(0, R)$. Assume that $y \in B(0, R)$, $\in > 0$, N is a positive integer and non overlapping intervals $[t_k, \omega_k] \subset J$, $k = 1, 2, \ldots, n$. Condition (A_3) that implies that

$$\begin{aligned} \sum_{k=1}^{n} |f_1(t_k, y(t_k)) - f_1(\omega_k, y(\omega_k))| &\leq \sum_{k=1}^{n} |f_1(t_k, y(\omega_k)) - f_1(\omega_k, y(\omega_k))| \\ &\quad + \sum_{k=1}^{n} |f_1(t_k, y(t_k)) - f_1(t_k, y(\omega_k))| \\ &\leq \sum_{k=1}^{n} |f_1(t_k, y(\omega_k)) - f_1(\omega_k, y(\omega_k))| \\ &\quad + \sum_{k=1}^{n} p_1(t_k) |y(t_k) - y(\omega_k)|. \end{aligned}$$

The same condition also implies that there are positive number δ_k, $k = 1, 2, \ldots, n$ such that for each $k = 1, 2, \ldots n$

$$|f_1(t_k, y(\omega_k)) - f_1(\omega_k, y(\omega_k))| < \in/2N, \text{ whenever } |t_k - \omega_k| < \delta_k . \tag{ii}$$

Denote $M_p = max\{\ p_1(t_k)\ |\ k = 1, \ldots, N\}$. Since x is absolutely continuous, there is a $\delta_0 > 0$ such that

$$\sum_{k=1}^{n} |y(t_k) \text{-} y(\omega_k)| < \in/2\, M_p, \text{ whenever } \sum_{k=1}^{n} |t_k - \omega_k| < \delta_0 . \tag{iii}$$

Choose $\delta = min\{\ \delta_k\ |\ k = 0, 1, \ldots, N\}$, then inequalities (i) - (iii) imply that

$$\sum_{k=1}^{n} |f_1(t_k, y(\omega_k)) - f_1(\omega_k, y(\omega_k))| < \epsilon, \text{ whenever } \sum_{k=1}^{n} |t_k - \omega_k| < \delta,$$

so that $f_1(t, y(t))$ is absolutely continuous whenever $y \in AC^1(J, \boldsymbol{R})$. Since $\mu'(t)$ is absolutely continuous, $f_1(t, x(\mu(t)))$ is absolutely continuous whenever $x \in AC^1(J, \boldsymbol{R})$. The proof that $f_2(t, x(\mu(t)))$ is absolutely continuous whenever $x \in AC^1(J, \boldsymbol{R})$ is similar.
Since g is a Caratheodory function and satisfies inequality (3.1),
then $g(s, x(\sigma(s)), x'(\eta(s))) \in L^1(J, \boldsymbol{R})$ whenever $x \in B(0, r)$.

Thus integral over [a,b] is

$\int G(t, s)\, g(s, x(\sigma(s)), x'(\eta(s)))\, ds \in AC^1(J, \boldsymbol{R})$ whenever $x \in B(0, r)$.

Define operators $A : X \to X$ and $T : B(0, r) \to X$ by

$$Ax(t) = f(t, x(\mu(t))), \ t \in J \tag{3.10}$$

and

$$Tx(t) = \int G(t, s)\, g(t, x(\sigma(s)), x'(\eta(s)))\, ds, \quad t \in J. \tag{3.11}$$

Now FBVP (1.1) and FIE (2.3) have the same solutions which also are the solution of the operator equation

$$Ax(t)Tx(t) = x(t), \quad t \in J. \tag{3.12}$$

We shall show that the operators A and T satisfy all the conditions of Theorem 2.2. First we show that the operator T is continuous and compact in $B(0, r)$. Let $(x_n)_{n=0}^{\infty}$ be a converging sequence in $B(0, r)$ such that

$$\lim_{n\to\infty} x_n = x$$

$$\text{i.e. } \max\{ \sup\{ |x(t) - x_n(t)| \mid t \in J \},$$
$$\sup\{ |x'(t) - x'_n(t)| \mid t \in J \} \} \to 0 \text{ when } n \to \infty.$$

Then by assumption (A_0) and (A_1) and Lebesgue Dominated convergence theorem

$$\lim Tx_n(t) = \int G(t, s)\, g(s, x_n(\sigma(s)), x'_n(\eta(s)))\, ds \quad n\to\infty$$
$$= \int G(t, s)\, g(s, x(\sigma(s)), x'(\eta(s)))\, ds = Tx(t), \qquad t \in J$$

and

$$\lim (Tx_n)'(t) = \int G_1(t, s)\, g(s, x_n(\sigma(s)), x'_n(\eta(s)))\, ds \ n\to\infty$$
$$= \int G_1(t, s)\, g(s, x(\sigma(s)), x'(\eta(s)))\, ds = (Tx)'(t), \qquad t \in J.$$

Thus T is continuous in $B(0, r)$.

Assume that y is an element of $B(0, r)$. Then $||y||_{AC^1} \leq r$ and by the condition (A_1) and inequalities (3.6),

$$|Ty(t)| \leq \int G(t, s)|\, g(s, y(\sigma(s)), y'(\eta(s)))|\, ds$$
$$\leq \int (b - a)\psi(s)\phi(\max\{ |y(\sigma(s))|, y'(\eta(s))| \})ds$$
$$\leq (b - a)||\psi||_{L^1}\, \phi(r) \quad t \in J.$$

Furthermore, $|(Ty)'(t)| \leq \int G_1(t, s)|\, g(s, y(\sigma(s)), y'(\eta(s)))|\, ds$

$\leq \int \psi(s)\phi(\max\{|y(\sigma(s))|, y'(\eta(s)))|\})ds$

$\leq ||\psi||_{L^1}\, \phi(r), \quad t \in J.$

Hence

$||Ty||_{AC^1} \leq M(r) = \max\{1, b-a\}||\psi||_{L^1}\, \phi(r)$, whenever $y \in B(0, r)$.

As a result every sequence $(Tx_n)_0^\infty$ is uniformly bounded when $(x_n)_0^\infty$ is a sequence of $B(0, r)$. We show next that the image $T[B(0, r)]$ of the closed ball $B(0, r)$ is equicontinuous. Let $x \in B(0, r)$ and $a \leq v \leq t \leq b$. Then by (3.5)

$$| Tx(t) - Tx(v)| \leq \int |G(t, s) - G(v, s)||g(s, x(\sigma(s)), x'(\eta(s)))|\, ds$$

$$\leq \int |G(t, s) - G(v, s)|\, \psi(s)\phi(\max\{|y(\sigma(s))|, y'(\eta(s)))|\})\, ds$$

$$\leq \int (t-v)\psi(s)\phi(r)\, ds = \phi(r)\, ||\psi||_{L^1}\,(t-v)$$

and

$$| (Tx)'(t) - (Tx)'(v)| \leq \int |G_1(t, s) - G_1(v, s)||g(s, x(\sigma(s)), x'(\eta(s)))|\, ds$$

$$\leq \int \psi(s)\phi(\max\{| y(\sigma(s))|, | y'(\eta(s)))|\})\, ds$$

$$= \phi(r) \int_v \psi(s)\, ds .$$

Hence

$$|| Tx(t) - Tx(v) ||_{AC1} \leq \phi(r)\max\{| t - v|\, ||\psi||_{L1}, | \int \psi(s)\, ds | \}$$

whenever $x \in B(0, r)$ and $t, v \in J$, where

$$\lim_{t \to v} \max\{ | t - v|\, ||\psi||_{L1}, | \int_v \psi(s)\, ds | \} = 0,$$

Since $\psi \in L^1(J, \mathbf{R}_+)$. As a result the set $T[B(0, r)]$ is equicontinuous in $AC^1(J, \mathbf{R})$, and consequently T is a compact operator on $B(0, r)$ by Arzela-Ascoli theorem.

We consider now the operator A defined by (3.10) and show first that $A[B(0, R)] \subseteq B(0, R/M)$. Suppose that $x \in B(0, R)$.

Then

$$| Ax(t) | = | f(t, x(\mu(t))) | \leq | f(t, x(\mu(t))) - f(t, 0)| + | f(t, 0)|$$

$$\leq p(t)\, | x(\mu(t)) | + | f(t, 0) | \leq M_0 R + c_0, \qquad \text{for all } t \in J,$$

where $M_0 = \sup\{ p(t) \mid t \in J \}$ and $c_0 = \sup\{ | f(t, 0)| \mid t \in J \}$, and

$$| (Ax)'(t) | \leq | f_1(t, x(\mu(t))) - f_1(t, 0)| + | f_1(t, 0)|$$

$$+ | \mu(t) ||\, x'(\mu(t)) | (f_2(t, x(\mu(t))) - f_2(t, 0)| + | f_2(t, 0)|$$

$$\leq p_1(t)\, | x(\mu(t))| + | f_1(t, 0)|$$

$$+ m_1 | x(\mu(t)) | (p_2(t)| x(\mu(t)) | + | f_2(t, 0)|)$$

$$\leq M_1 R + c_1 + m_1 R\, (M_2 R + c_2)$$

$$= m_1 M_2 R^2 + (M_1 + m_1 c_2)R + c_1, \qquad \text{for all } t \in J,$$

where $m_1 = \sup_{t \in J} |\mu'(t)|$, $M_1 = \sup\{ p_1(t) | t \in J \}$, $M_2 = \sup\{ p_2(t) | t \in J \}$, $c_1 = \sup\{ | f_1(t, 0)| \mid t \in J \}$ and $c_2 = \sup\{ | f_2(t, 0)| \mid t \in J \}$.

According to inequality (3.8)

$$\max\{ M_0 R + c_0,\ m_1 M_2 R^2 + (M_1 + m_1 c_2)R + c_1 \} \leq R/M,$$

so that

$$\|Ax\|_{AC}{}^{1} = \max\{\sup_{t\in J} |Ax(t)|, \sup_{t\in J} |(Ax)'(t)|\} \le R/M \text{ whenever } x \in B(0, R).$$

Assume that $x, y \in B(0, R)$. Then by conditions (A_2) and (A_3)

$$\begin{aligned} | Ax(t) - Ay(t)| &= | f(t, x(\mu(t))) - f(t, y(\mu(t))) | \\ &\le p(t) \,| x(\mu(t)) - y(\mu(t))|, \quad \text{for each } t \in J, \end{aligned}$$

so that

$$| Ax(t) - Ay(t)| \le M_0 \| x\text{-}y \|_{AC}{}^{1}, \qquad \text{for each } t \in J,. \tag{a}$$

Moreover, by the same conditions

$$\begin{aligned} | (Ax)'(t) - (Ay)'(t) | &\le | f_1(t, x(\mu(t))) - f_1(t, y(\mu(t))) | \\ &\quad + |\mu'(t)| \,| x'(\mu(t)) f_2(t, x(\mu(t))) - y'(\mu(t)) f_2(t, y(\mu(t)))| \\ &\le p_1(t) \,| x(\mu(t)) - y(\mu(t))| \\ &\quad + m_1 (| x'(\mu(t))| \,| f_2(t, x(\mu(t))) - f_2(t, y(\mu(t)))| \\ &\quad + | f_2(t, y(\mu(t)))| \,| x'(\mu(t)) - y'(\mu(t))|) \\ &\le M_1 | x(\mu(t)) - y(\mu(t))| \\ &\quad + m_1 (M_2R \,| x(\mu(t)) - y(\mu(t)) | \\ &\quad + (M_2R+c_2) \,| x'(\mu(t)) - y'(\mu(t))|), \text{ for each } t \in J, \end{aligned}$$

so that

$$\begin{aligned} | Ax'(t) - Ay'(t)| &\le (M_1 + m_1M_2R) | x(\mu(t)) - y(\mu(t)) | \\ &\quad + m_1 (M_2R+c_2) \,| x'(\mu(t)) - y'(\mu(t))|) \qquad \text{(b)} \\ &\le K_0 \| x\text{-}y \|_{AC}{}^{1}, \qquad \text{for each } t \in J, \end{aligned}$$

where $K_0 = \max \{M_1 + m_1M_2R, m_1c_2 + m_1M_2R \} = \max \{ M_1, m_1c_2 \} + m_1M_2R$.
Inequalities (a) and (b) imply that

$$\|Ax - Ay\| \le \alpha \| x\text{-}y \|_{AC}{}^{1} \quad \text{whenever } x, y \in B(0, R),$$

where $\alpha = \max \{M_0, K_0\} = \max \{ M_0, \max \{ M_1, m_1c_2 \} + m_1M_2R\}$. Hence A is a Lipschitz operator in $B(0, R)$ with Lipschitz constant α, and by inequality (3.7) $\alpha M < 1$.

Thus the conditions (a) and (b) of Proposition 2.1 are satisfied ($r_1 = R$ and $r_2 = r$), and hence if $R \le r$ then operator equation $x = Ax\, Tx$ has a solution in $B(0, R)$. Otherwise either its conclusion (i) or (ii) holds. We show that conclusion (ii) is impossible.

Let u be a solution of the operator equation

$$\mu(t) = \lambda A ((1/\lambda) u(t)) Tu(t), \quad t \in J,$$

with $\|u\|_{AC}{}^{1} = r$ for some $0 < \lambda < 1$. Then $r \le \lambda R$ and inequalities (1) and (2) imply that

$$| \mu(t)| \le \lambda\, f \,| (t, (1/\lambda) u(u(t))) | \int G(t, s)| \, g(s, u(\sigma(s)), u'(\eta(s)))| \, ds$$

$$\le \lambda \quad f \,| t, \quad (1/\lambda) \; u(u(t)) \; - f(t, 0) \; + | f(t, 0)| \; (b - a) \|\psi\|_{L}{}^{1} \, \phi(r)$$

$$\le [M_0 \,| u(\mu(t)) | + \lambda c_0] (b\text{-}a) \|\psi\|_{L}{}^{1} \, \phi(r), \quad t \in J, \tag{c}$$

Similarly for,

where $M_0 = \sup \{ p(t) \mid t \in J \}$ and $c_0 = \sup_{t\in J} | f(t, 0) |$, and

$$\le [M_1 \,| u(\mu(t))| + \lambda c_1 + | u'(\mu(t)) | \,| \mu'(t)) | (M_2R + c_2)]$$

$(b- a) \|\psi\|_{L^1} \phi(r)$
$+ [M_0 | u(\mu(t)) | + \lambda c_0] \|\psi\|_{L^1} \phi(r), t \in J,$ (d)

where $M_1 = sup\{ p_1(t)| t\in J \}$, $M_2 = sup \{ p_2(t) | t\in J \}$, $c_1 = sup_{t\in J} | f_1 (t, 0) |$ and $c_2 = sup_{t\in J} | f_2(t, 0) |$. Inequalities (c) and (d) imply that

$$\sup_{t\in J} |\mu(t)| \leq (b - a) \|\psi\|_{L^1} \phi(r) [M_0 r + \lambda c_0] \text{ and}$$

$$\sup |\mu'(t)| \leq (b - a) \|\psi\|_{L^1} \phi(r) [M_1 r + \lambda c_1 + m_1 (M_2R + c_2)r]$$

$$+ \|\psi\|_{L^1} \phi(r) [M_0 r + \lambda c_0],$$

thus

$$r \leq \frac{b-a}{\max\{1, b-a\}} M(r) \max \left\{ \begin{array}{l} M_0 r + c_0, \left[\frac{M_0}{b-a} + M_1 + m_1 (M_2 R + c_2) r \right] \\ + \frac{c_0}{b-a} + c_1 \end{array} \right\}$$

since $r = ||u||_{AC^1}$ and $0 < \lambda < 1$. This is a contradiction to (3.9), and hence the conclusion (ii) is not valid. Consequently, the conclusion (i) is valid, and the FBVP (1.1) has a solution in $B(0, r)$.

Remark 3.1 : The FBVP (1.1) has a nonzero solution if all the conditions of Theorem 3.1 are satisfied and there exists a subset I of the interval J such that means$(I) > 0$ and $g(s, 0, 0) \neq 0$ whenever $s \in I$.

Proposition 3.1. *Assume that there exist real numbers* $r > 0$ *and* $R > 0$ *such that the hypothesis* $(A_0) – (A_2)$ *hold,* f *is only function of variable* x *i.e.* $f(t, x) = q(x) \neq 0$, *and* (A'_3)

$$| q(x) - q(y) | \leq a_0 | x – y | \text{ and}$$
$$| q'(x) - q'(y) | \leq a_1 | x – y | \text{ whenever } | x |, | y| \leq R, \quad (3.14)$$

where a_0 *and* a_1 *are positive constants.*

Suppose that

$$M(r) \max \{ a_0 , m_1(M_0 R+ q_1)\} < 1 \quad (3.15)$$

Where $M(r) = \max \{1, b- a \} \|\psi\|_{L^1} \phi(r)$, $q_0 = |q (0)|$ *and* $q_1 = |q'(0)|$, *and*

$$\max \{ a_0 R+ q_0 , m_1R (a_1 R+ q_1)\} < R/M \quad (3.16)$$

and

$$r > \frac{(b – a)}{\max \{1, b- a\}} M(r) \max \left\{ a_0 r+q_0 , m_1 r (a_1R + q_1)+ \frac{a_0 r+ q_0}{b - a} \right\} \quad (3.17)$$

Then FBVP (1.1) has a solution $x \in AC^1(J, \mathbf{R})$ *with* $||x||_{AC^1} \leq r$.

Proof : Since operator T is the same as in the proof of Theorem 3.1 we only need to show that operator $Ax(t) = q(x(\mu(t)))$ maps every function $x \in B(0, R)$ into the ball $B(0, R/M)$ condition (b) of Proposition 2.1 is valid.

In the same way as in the proof of Proposition 3.1 we can show that the derivative $\mu'(t)\, x'(\mu(t))\, q'(x(\mu(t)))$ is absolutely continuous whenever $x \in B(0, R)$. Now assume that $x \in B(0, R)$. Then

$$| Ax(t) | = | q(x(\mu(t))) | \leq | q(x(\mu(t))) - q(0)| + | q(0) | \leq a_0 | x(\mu(t)) | + | q(0)| \leq a_0 R + q_0$$

where $q_0 = | q(0) |$ and

$$|(Ax)'(t)| = | \mu'(t)\, x'(\mu(t))\, q'(x(\mu(t))) | \leq | \mu'(t)| \, | x'(\mu(t)) | \, (| q'(x(\mu(t))) - q'(0)| + | q'(0) |) \leq m_1 R(a_1 R + q_1),$$

where $q_1 = | q'(0) |$, so that by inequality (3.16)

$$|| Ax ||_{AC^1} \leq max \{ a_0 R + q_0 , m_1 R (a_1 R + q_1)\} \leq R/M,$$

whenever $x \in B(0, R)$.

Moreover, if $x, y \in B(0, R)$, then

$$| Ax(t) - Ay(t) | = | q(x(\mu(t))) - q(y(\mu(t))) | \leq a_0 | x(\mu(t)) - y(\mu(t))|$$

and

$$|(Ax)'(t) - (Ay)'(t)| = |\mu'(t) x'(\mu(t))\, q'(x(\mu(t))) - \mu'(t)\, y'(\mu(t)) q'(y(\mu(t))) |$$
$$\leq m_1 (| x'(\mu(t)) | \, | q'(x(\mu(t))) - q'(y(\mu(t))) | + | q'(y(\mu(t)))| \, | x'(\mu(t)) - y'(\mu(t)) |)$$
$$\leq m_1 (a_1 R \, | x(\mu(t)) - y(\mu(t)) | + (a_1 R + q_1) \, | x'(\mu(t)) - y'(\mu(t)) |)$$

hence

$$|| Ax - Ay ||_{AC^1} \leq max \{ a_0 , m_1 (a_1 R + q_1)\} \, || x - y ||_{AC^1}$$

Choose $\alpha = max \{ a_0 , m_1 (a_1 R + q_1)\}$, then A is a Lipschitz operator in $B(0, R)$ and inequality (3.15) implies that $\alpha M < 1$. Now by Proposition 2.1 FBVP (1.1) has a solution $x \in AC^1(J, \mathbf{R})$ with $|| x ||_{AC^1} \leq R$, if $R \leq r$. Otherwise conclusion (i) or (ii) of Proposition 2.1 is valid. Suppose u is a solution of the operator equation

$$u(t) = \lambda A ((1/\lambda)\, u(t))\, Tu(t), \qquad t \in J.$$

The same way as in Theorem 3.1 we can show that

$$\sup_{t \in J} | u(t) | \leq max \{1, b - a \} \, ||\psi||_{L^1} \phi(r) \, [a_0 r + q_0]$$

and

$$\sup_{t \in J} | u'(t) | \leq max \{1, b - a\} \, ||\psi||_{L^1} \phi(r) \, [m_1 r(a_1 r + q_1)] + ||\psi||_{L^1} \phi(r) \, [a_0 r + q_0] ,$$

thus

$$r \leq \frac{(b-a)}{\max\{1,\, b-a\}} M(r) \max \left\{ a_0 r + q_0,\ m_1 r(a_1 R + q_1) + \frac{a_0 r + q_0}{b-a} \right\}.$$

This is a contradiction to (3.17), so that the FBVP (1.1) has a solution in $B(0, R)$. Hence proved.

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