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REMARKS ON IDEAL ON DISTRIBUTIVE Q-LATTICES

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Abstract- In this paper, we prove if J be an ideal of distributive q-lattice A then for any prime ideal P containing J, J (P) is an ideal of A such that J⊆J (P) ⊆P also if P be a prime ideal containing an ideal J of distributive q-lattice A and Q be a prime ideal such that J ⊆ Q ⊆ P then J (P) ⊆ Q. We define a relation Θ_J on A and prove is congruence relation on A under condition. and for f: $A_1 \rightarrow A_2$ be a homomorphism we prove If J is an ideal of A_2 then $f^{-1}(J)$ } is an ideal of A_1 and similar for filter

INTRODUCTION

Ivan Chajda [2] introduced the concept of a q-lattice and defined distributive q-lattice. After that G. C. Rao, P. Sundarayya, S. Kalesha vali, and Ravi Kumar Bandaru [1], they defined ideals of a distributive q-lattice. They proved if A be a distributive q-lattice then I (A), the set of all ideals of A is a lattice under set inclusion. They give some equivalent conditions of a distributive q-lattice to become a distributive lattice in terms of ideals. In paper Filter and Annihilator in Distributive q-lattices, A. D. Lokhande, Ashok S Kulkarni [4], define Filter in a distributive q-lattice and we prove if A be a distributive q-lattice then F (A), the set of all filters of A is a lattice under set inclusion. Also we derive some properties of filter. We also give some equivalent conditions of a distributive q-lattice to become distributive lattice in terms of filter. G. C. Rao and M. Sambasiva Rao [5] defined ‘ annihilator ’ in Almost Distributive Lattice (ADL_s) and derived some properties, In paper [4] we define annihilator in distributive q-lattice A and prove for any ideal I of distributive q-lattice A and a∈ A , the annihilator (a:I) is an ideal of A and derive some properties. In this paper we define J(P) and prove if J be an ideal of distributive q-lattice A then for any prime ideal P containing J, J(P) is an ideal of A such that $J \subseteq J(P) \subseteq P$ also if P be a prime ideal containing an ideal J of distributive q-lattice A and Q be a prime ideal such that $J \subseteq Q \subseteq P$ then $J(P) \subseteq Q$. We define a relation Θ_J on A and prove is congruence relation on A under some condition. and for $f: A_1 \rightarrow A_2$ be a homomorphism we prove If J is an ideal of A_2 then $f^{-1}(J)$ } is an ideal of A_1 and similar for filter

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PRELIMINARIES

Some of the following definitions and results are taken from [1] and [4]

Definition II.1: An algebra (A, ∨, ∧) whose binary operations ∨, ∧ satisfy the following is called a q-lattice.

(i) $a \vee b = b \vee a$; $a \wedge b = b \wedge a$ (commutativity)

(ii) $a \vee (b \vee c) = (a \vee b) \vee c$; $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ (associatativity)

(iii) $a \vee (a \wedge b) = a \vee a$; $a \wedge (a \vee b) = a \wedge a$ (weak- absorption)

(iv) $a \vee b = a \vee (b \vee b)$; $a \wedge b = a \wedge (b \wedge b)$ (weak- idempotence)

(v) $a \vee a = a \wedge a$ (equalization)

Definition II.2: A q-lattice (A, ∨, ∧) is distributive if it satisfies the identity

$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z \in A$

Lemma II.I : Let A be a distributive q-lattice then the following identity hold

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in A$.

Definition II.3: Ideal of a distributive q-lattice:

A nonempty subset I of a distributive q-lattice A is called an ideal of A if

i) x, y I $\Rightarrow x \vee y \in I$

ii) $x \in I$ and $a \in A \Rightarrow a \wedge x \in I$

Definition II.4: Let L be an ADL and J an ideal of L .Then for any non empty subset S of L , define the annihilator (S:J) of A relative to J as follows :

(S: J) = {x∈ L/ x∧ a∈ J for all a ∈ S}

If S = {a}, then ({a}: J) is abbreviated as the annihilator (a: J) of a∈ L

Definition II.5: - Filter of a distributive q- lattice :

A nonempty subset F of a distributive q-lattice A is called a filter of A, if.

i) x, y∈ F ⇒ x ∧ y ∈ F
ii) x∈ F and a∈ A ⇒ a ∨ x ∈ F

Definition II.6 :

Let A be distributive q-lattice and I is an ideal of A then for any non empty subset P of A define annihilator

P* = (P: I) = {x ∈ A / x ∧ a ∈ I, for all a ∈ P}

If P = {a} then we write ({a}: J) as the annihilator (a: J) of a ∈ P

Therefore (a: I)={x ∈A / x ∧ a ∈ I , for all a ∈ {a}}

PRIME IDEAL J (P):

Definition: III.1:

A proper ideal I of a distributive q-lattice A is called a prime ideal if for any x ,

y ∈ A , x ∧ y ∈ A implies x ∈ A or y ∈ A

Definition: III.2:

Let P be a prime ideal containing an ideal J of distributive q-lattice A then define J(P) = { x ∈ A / x ∧ t ∈ J for some t ∉ P}∈

= { x ∈ A / t ∧ x ∈ J for some t ∉ P}

Lemma:III.1

Let J be an ideal of distributive q-lattice A then for any prime ideal P containing J, J(P) is an ideal of A such that $J \subseteq J(P) \subseteq P$

Proof : Let P be a prime ideal containing an ideal J in A

Let $x \in J$, then as J is an ideal therefore $a \wedge x \in J$ for any $a \in A$

i.e. $a \wedge x = x \wedge a \in J$ for some $a \notin P$ also

this implies $x \in J(P)$ therefore $J \subseteq J(P)$

Now to show J(P) is an ideal:

Let a, b $\in$ J(P) implies $x \wedge a = a \wedge x \in J$ and $y \wedge b = b \wedge y \in J$ for some $x \notin P$ and $y \notin P$

Consider$(x \wedge y) \wedge (a \vee b) = ((x \wedge y) \wedge a) \vee ((x \wedge y) \wedge b)$

Now as P is prime ideal and $x \notin P$, $y \notin P$ implies $x \wedge y \notin P$

$= (x \wedge (y \wedge a)) \vee (x \wedge (y \wedge b))$

$= (x \wedge (a \wedge y)) \vee (x \wedge (y \wedge b))$

$= ((x \wedge a) \wedge y) \vee (x \wedge (y \wedge b)$

As $x \wedge a \in J$, $y \in A$

Implies $(x \wedge a) \wedge y \in J$

And $y \wedge b \in J$, $x \in A$

Implies $x \wedge (y \wedge b) \in J$

And as J is an ideal

Implies $((x \wedge a) \wedge y) \vee (x \wedge (y \wedge b)) \in J$

Implies $(x \wedge y) \wedge (a \vee b) \in J$ where $x \wedge y \notin P$

Implies $a \vee b \in J(P)$

Now let $a \in J(P)$ and $r \in A$

Then $x \wedge a \in J$ for some $x \notin P$

Now to show $r \wedge a \in J(P)$

Consider $x \wedge (r \wedge a) = x \wedge (a \wedge r)$

$=(x \wedge a) \wedge r$

As $x \wedge a \in J$ and $r \in A$ and J is an ideal so $(x \wedge a) \wedge r \in J$

Implies $x \wedge (r \wedge a) \in J$ and $x \notin P$

Implies $r \wedge a \in J(P)$

Therefore J(P) is an ideal of A such that $J \subseteq J(P)$

Now let $a \in J(P)$

Then $x \wedge a \in J$ for some $x \notin P$

Implies $x \wedge a \in P$ since $J \subseteq P$

And since P is prime ideal and $x \notin P$

Implies $a \in P$

Implies $J(P) \subseteq P$

Theorem:III.1:

Let P be a prime ideal containing an ideal J of distributive q-lattice A and Q be a prime ideal such that $J \subseteq Q \subseteq P$ then $J(P) \subseteq Q$

Proof: We know $J(P) = \{ x \in A / x \wedge t \in J , \text{ for some } t \notin P\}$

$$= \{ x \in A / t \wedge x \in J, \text{ for some } t \notin P\}$$

By previous lemma $J \subseteq J(P) \subseteq P$

Let Q be a prime ideal such that $J \subseteq Q \subseteq P$

Now let $x \in J(P)$

Implies $t \wedge x \in J \subseteq Q$ for some $t \notin P$

Suppose $x \notin Q$

Since Q is prime, we get $t \in Q \subseteq P$

Which is contradiction to $t \notin P$

Thus $x \in Q$

Implies $J(P) \subseteq Q$

Defination:III.3:

Let A be distributive q- lattice. Two ideals I , J of A are called co-maximal

If $I \cup J = A$

Theorem:III.2:

Let A be distributive q-lattice then for any ideal J of A , then

Statements (a) ⇒ (b) ⇒(c)⇒(d)⇒ (e)

(a)any two distinct minimal prime ideals belonging to J are co-maximal

(b)every prime ideal containing J ,contains a unique minimal prime ideal belonging to J

(c) For each prime ideal P containing J $(J \subseteq P)$ J(p) is a prime ideal.

(d) for any $a, b \in A$, $a \wedge b \in J$ implies

$(a:J) \cup (b:J) = A$

(e) For any $a, b \in A$, $(a \vee b: J) \subseteq (a : J) \cup (b :J)$

Proof: (a) implies (b)

Let P be a prime ideal of A that is containing J $(J \subseteq P)$ Suppose P contains two distinct minimal prime ideals Q_1 and Q_2 both belonging to J

Hence by a)

A= $Q_1 \cup Q_2 \subseteq P$

Which is a contradiction hence prime ideal P containing J ,contains a unique minimal prime ideal belonging to J

(b) implies (c)

Assume that every prime ideal containing J (say J⊆P) contains a unique minimal prime ideal belonging to J

Let P be a prime ideal containing J. since J(P) ⊆P that is J ⊆J(P) ⊆P

We get that J(P) is the unique minimal prime ideal belonging to J and contained in P therefore J(P) will be a prime ideal.

(c) implies (d)

Assume for each prime ideal P containing J that is J⊆P, J(P) is a prime ideal.

Let a, b ∈ A such that $a \wedge b \in J$

Suppose (a:J) J

Suppose (a:J)∪ (b:J) ≠ A

Then there exists a prime ideal P containing J such that (a:J)∪ (b:J) ⊆P

Hence (a:J) ⊆P and (b:J) ⊆P

That is if x ∈ (a:J) implies x ∈ P and x ∈ (b:J) implies x ∈ P that is $x \wedge a \in J$ implies x ∈ P and $x \wedge b \in J$ implies x ∈ P

Implies a ∉ J(P) and b ∉ J(P)

Since J(P) is prime ideal

We get $a \wedge b \notin J(P)$

Which is contradiction to that $a \wedge b \in J$ since J ⊆ J(P)

Hence (a:J)∪ (b:J) =A

(d) implies (e)

Assume for any a, b ∈ A , a ∧ b ∈ J implies

(a:J) ∪ (b:J) =A

Let w ∈ (a v b :J) then w ∧ (a v b) ∈ J

That is (w∧ a) v (w∧ b) ∈ J therefore from condition (d)

(w ∧ a : J) ∪ (w ∧ b : J) = A

Now w ∈ A implies w ∈ (w∧ a : J) or w ∈ (w∧ b : J)

Implies w ∧ (w ∧ a) ∈ J or w ∧ (w ∧ b) ∈ J

Implies (w ∧ a) ∈ J or (w ∧ b) ∈ J

Implies w ∈ (a : J) or w ∈ (b : J)

Implies w ∈ (a : J) ∪ (b : J)

Implies (a v b: J) ⊆ (a : J) ∪ (b :J)

Theorem:III.4:

Let A be distributive q-lattice. Define a relation Θ_J if and only if (a :J) = (b:J) then Θ_J is a equivalence relation on A and J is congruence relation on A under condition (a v c: J) ⊆ (a :j) ∩ (c : J)

Proof : i)As (a:J) = (a:J) implies (a , a) ∈ Θ_J for all a ∈ A

Therefore Θ_J is reflexive

ii) If (a, b) ∈ Θ_J implies (a:J) = (b:J) implies (b:J) = (a:J)

Implies (b, a) ∈ Θ_J therefore relation Θ_J is symmetric

iii) If (a, b) ∈ Θ_J , (b, c) ∈ Θ_J

Implies (a:J) = (b:J) , (b:J) = (c:J)

Implies (a:J) = (b:J) = (c:J)

Implies (a:J) = (c:J)

Implies (a ,c) $\in \Theta_J$ therefore relation Θ_J is transitive

Therefore relation Θ_J is an equivalence relation on A

Now let (a, b) , (c ,d) $\in \Theta_J$

Implies (a:J) = (b:J) and (c:J) = (d :J)

Now, let $x \in (a \wedge c : J)$

$\Leftrightarrow x \wedge (a \wedge c) \in J$

$\Leftrightarrow (x \wedge a) \wedge c \in J$

$\Leftrightarrow x \wedge a \in (c : J)$

$\Leftrightarrow x \wedge a \in (d : J)$

$\Leftrightarrow (x \wedge a) \wedge d \in J$

$\Leftrightarrow x \wedge (a \wedge d) \in J$

$\Leftrightarrow x \wedge (d \wedge a) \in J$

$\Leftrightarrow (x \wedge d) \wedge a \in J$

$\Leftrightarrow x \wedge d \in (a : J)$

$\Leftrightarrow x \wedge d \in (b : J)$

$\Leftrightarrow (x \wedge d) \wedge b \in J$

$\Leftrightarrow x \wedge (d \wedge b) \in J$

$\Leftrightarrow x \wedge (b \wedge d) \in J$

$\Leftrightarrow$ $x \in (b \wedge d : J)$

Hence $(a \wedge c : J) = (b \wedge d : J)$

Therefore $(a \wedge c , b \wedge d) \in \Theta_J$

Now to show $(a \vee c , b \vee d) \in \Theta_J$

Let $x \in (a : J) \cap (c : J)$

Implies $x \in (a : J)$ and $x \in (c : J)$

Implies $x \wedge a \in J$ and $x \wedge c \in J$ and as J is an ideal

Implies $(x \wedge a) \vee (x \wedge c) \in J$

Implies $x \wedge (a \vee c) \in J$

Implies $x \in (a \vee c : J)$

Hence $(a : J) \cap (c : J) \subseteq (a \vee c : J)$ and from given condition

$(a \vee c : J) \subseteq (a : j) \cap (c : J)$

Therefore $(a \vee c : J) = (a : j) \cap (c : J)$

Implies $(a \vee c : J) = (b : j) \cap (d : J)$

Implies $(a \vee c : J) = (b \vee d : j)$

Hence $(a \vee c , b \vee d) \in \Theta_J$

Theorem:III.5:

Let A_1 and A_2 be two distributive q-lattices and $f: A_1 \rightarrow A_2$ be a homomorphism then

1) If J is an ideal of A_2 then $f^{-1}(J) = \{ x \in A_1 / f(x) \in J \subseteq A_2 \}$ is an ideal of A_1

2) If J is filter of A_2 then $f^{-1}(J) = \{ x \in A_1 / f(x) \in J \subseteq A_2 \}$ is filter of A_1

Proof: Let $f: A_1 \rightarrow A_2$ be a homomorphism

1) Let J is an ideal of A_2 and $f^{-1} = \{ x \in A_1 / f(x) \in J \subseteq A_2 \}$

Let $x , y \in f^{-1}(J)$ then $f(x) , f(y) \in J$

As J is an ideal , $f(x) \vee f(y) = f(x \vee y) \in J$

Therefore $x \vee y \in f^{-1}(J)$

Now if $x \in f^{-1}(J)$ and $r \in A_1$

Then $f(x) \in J$, $f(r) \in A_2$ and as J is an ideal of A_2 , therefore

$f(x) \wedge f(r) = f(x \wedge r) \in J$

Thus $x \wedge r \in f^{-1}(J)$ therefore $f^{-1}(J) = \{ x \in A1 / f(x) \in J \subseteq A_2 \}$ is an ideal of A_1

2) Let J is filter of A_2 and $f^{-1}(J) = \{ x \in A_1 / f(x) \in J \subseteq A_2 \}$

Let $x , y \in f^{-1}(J)$ then $f(x) , f(y) \in J$

As J is filter , $f(x) \wedge f(y) = f(x \wedge y) \in J$

Therefore $x \wedge y \in f^{-1}(J)$

Now if $x \in f^{-1}(J)$ and $r \in A_1$

Then $f(x) \in J$, $f(r) \in A_2$ and as J is filter of A_2 ,therefore

$f(x) \vee f(r) = f(x \vee r) \in J$

Thus $x \vee r \in f^{-1}(J)$ therefore $f^{-1}(J) = \{ x \in A1 / f(x) \in J \subseteq A_2 \}$ is filter of A_1

Theorem: III.6:

Let A_1 and A_2 be two distributive q-lattices and $f: A_1 \rightarrow A_2$ be a homomorphism then

1) $f(P^*) \subseteq (f(P))^*$

Proof: let $x \in f(P:I)$

Implies there exits $y \in (P:I)$ such that $f(y) = x \in f(P:I)$

Implies $y \wedge a \in I$ for all $a \in P$

Implies $f(y \wedge a) \in f(I)$ for all $a \in P$

Implies $f(y) \wedge f(a) \in f(I)$ for all $a \in P$ that is for all $f(a) \in f(P)$

Implies $f(y) \in \{ f(P) : f(I)\}$

Implies $x \in \{ f(P) : f(I)\}$

Implies $f(P:I) \subseteq \{ f(P) : f(I)\}$

Therefore $f(P^*) \subseteq (f(P))^*$

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