

REMARKS ON IDEAL ON DISTRIBUTIVE Q-LATTICES

Ashok S. Kulkarni¹, A. D. Lokhande²

¹Research Student, D.A.B.N. College, Chikhali, Tal-Shirala, Dist-Sangli, Maharashtra, India

²Research Guide, HOD, Mathematics, Y. C. Warana Mahavidyalay Warananagar, Dist-Kolhapur, Maharashtra, India

Abstract- In this paper, we prove if J be an ideal of distributive q -lattice A then for any prime ideal P containing J , $J(P)$ is an ideal of A such that $J \subseteq J(P) \subseteq P$ also if P be a prime ideal containing an ideal J of distributive q -lattice A and Q be a prime ideal such that $J \subseteq Q \subseteq P$ then $J(P) \subseteq Q$. We define a relation Θ_J on A and prove is congruence relation on A under condition. and for $f: A_1 \rightarrow A_2$ be a homomorphism we prove If J is an ideal of A_2 then $f^{-1}(J)$ is an ideal of A_1 and similar for filter

INTRODUCTION

Ivan Chajda [2] introduced the concept of a q -lattice and defined distributive q -lattice. After that G. C. Rao, P. Sundarayya, S. Kalesha vali, and Ravi Kumar Bandaru [1], they defined ideals of a distributive q -lattice. They proved if A be a distributive q -lattice then $I(A)$, the set of all ideals of A is a lattice under set inclusion. They give some equivalent conditions of a distributive q -lattice to become a distributive lattice in terms of ideals. In paper Filter and Annihilator in Distributive q -lattices, A. D. Lokhande, Ashok S Kulkarni [4], define Filter in a distributive q -lattice and we prove if A be a distributive q -lattice then $F(A)$, the set of all filters of A is a lattice under set inclusion. Also we derive some properties of filter. We also give some equivalent conditions of a distributive q -lattice to become distributive lattice in terms of filter. G. C. Rao and M. Sambasiva Rao [5] defined ‘ annihilator ’ in Almost Distributive Lattice (ADL_s) and derived some properties, In paper [4] we define annihilator in distributive q -lattice A and prove for any ideal I of distributive q -lattice A and $a \in A$, the annihilator $(a:I)$ is an ideal of A and derive some properties. In this paper we define $J(P)$ and prove if J be an ideal of distributive q -lattice A then

for any prime ideal P containing J , $J(P)$ is an ideal of A such that $J \subseteq J(P) \subseteq P$ also if P be a prime ideal containing an ideal J of distributive q -lattice A and Q be a prime ideal such that $J \subseteq Q \subseteq P$ then $J(P) \subseteq Q$. We define a relation Θ_J on A and prove is congruence relation on A under some condition. and for $f: A_1 \rightarrow A_2$ be a homomorphism we prove If J is an ideal of A_2 then $f^{-1}(J)$ is an ideal of A_1 and similar for filter

PRELIMINARIES

Some of the following definitions and results are taken from [1] and [4]

Definition II.1: An algebra (A, \vee, \wedge) whose binary operations \vee, \wedge satisfy the following is called a q -lattice.

- (i) $a \vee b = b \vee a$; $a \wedge b = b \wedge a$ (commutativity)
- (ii) $a \vee (b \vee c) = (a \vee b) \vee c$; $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ (associativity)
- (iii) $a \vee (a \wedge b) = a \vee a$; $a \wedge (a \vee b) = a \wedge a$ (weak- absorption)
- (iv) $a \vee b = a \vee (b \vee b)$; $a \wedge b = a \wedge (b \wedge b)$ (weak- idempotence)
- (v) $a \vee a = a \wedge a$ (equalization)

Definition II.2: A q -lattice (A, \vee, \wedge) is distributive if it satisfies the identity

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \quad \text{for all } x, y, z \in A$$

Lemma II.1 : Let A be a distributive q -lattice then the following identity hold

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad \text{for all } a, b, c \in A.$$

Definition II.3: Ideal of a distributive q -lattice:

A nonempty subset I of a distributive q -lattice A is called an ideal of A if

- i) $x, y \in I \Rightarrow x \vee y \in I$
- ii) $x \in I$ and $a \in A \Rightarrow a \wedge x \in I$

Definition II.4: Let L be an ADL and J an ideal of L . Then for any non empty subset S of L , define the annihilator $(S:J)$ of A relative to J as follows :

$$(S: J) = \{x \in L / x \wedge a \in J \text{ for all } a \in S\}$$

If $S = \{a\}$, then $(\{a\}: J)$ is abbreviated as the annihilator $(a: J)$ of $a \in L$

Definition II.5: - Filter of a distributive q- lattice :

A nonempty subset F of a distributive q-lattice A is called a filter of A , if.

- i) $x, y \in F \Rightarrow x \wedge y \in F$
- ii) $x \in F \text{ and } a \in A \Rightarrow a \vee x \in F$

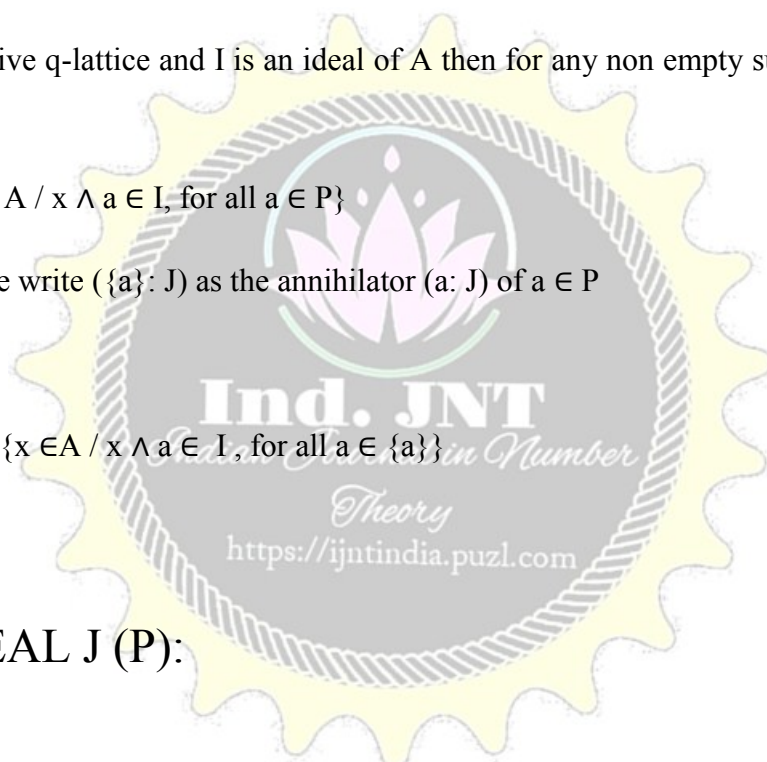
Definition II.6 :

Let A be distributive q-lattice and I is an ideal of A then for any non empty subset P of A define annihilator

$$P^* = (P: I) = \{x \in A / x \wedge a \in I, \text{ for all } a \in P\}$$

If $P = \{a\}$ then we write $(\{a\}: J)$ as the annihilator $(a: J)$ of $a \in P$

Therefore $(a: I) = \{x \in A / x \wedge a \in I, \text{ for all } a \in \{a\}\}$



PRIME IDEAL J (P):

Definition: III.1:

A proper ideal I of a distributive q-lattice A is called a prime ideal if for any $x,$

$y \in A, x \wedge y \in I$ implies $x \in I$ or $y \in I$

Definition: III.2:

Let P be a prime ideal containing an ideal J of distributive q-lattice A then define $J(P) = \{ x \in A / x \wedge t \in J \text{ for some } t \notin P\}$

$$= \{ x \in A / t \wedge x \in J \text{ for some } t \notin P\}$$

Lemma:III.1

Let J be an ideal of distributive q -lattice A then for any prime ideal P containing J , $J(P)$ is an ideal of A such that $J \subseteq J(P) \subseteq P$

Proof : Let P be a prime ideal containing an ideal J in A

Let $x \in J$, then as J is an ideal therefore $a \wedge x \in J$ for any $a \in A$

i.e. $a \wedge x = x \wedge a \in J$ for some $a \notin P$ also

this implies $x \in J(P)$ therefore $J \subseteq J(P)$

Now to show $J(P)$ is an ideal:

Let $a, b \in J(P)$ implies $x \wedge a = a \wedge x \in J$ and $y \wedge b = b \wedge y \in J$ for some $x \notin P$ and $y \notin P$

Consider $(x \wedge y) \wedge (a \vee b) = ((x \wedge y) \wedge a) \vee ((x \wedge y) \wedge b)$

Now as P is prime ideal and $x \notin P, y \notin P$ implies $x \wedge y \notin P$

$$= (x \wedge (y \wedge a)) \vee (x \wedge (y \wedge b))$$

$$= (x \wedge (a \wedge y)) \vee (x \wedge (y \wedge b))$$

$$= ((x \wedge a) \wedge y) \vee (x \wedge (y \wedge b))$$

As $x \wedge a \in J, y \in A$

Implies $(x \wedge a) \wedge y \in J$

And $y \wedge b \in J, x \in A$

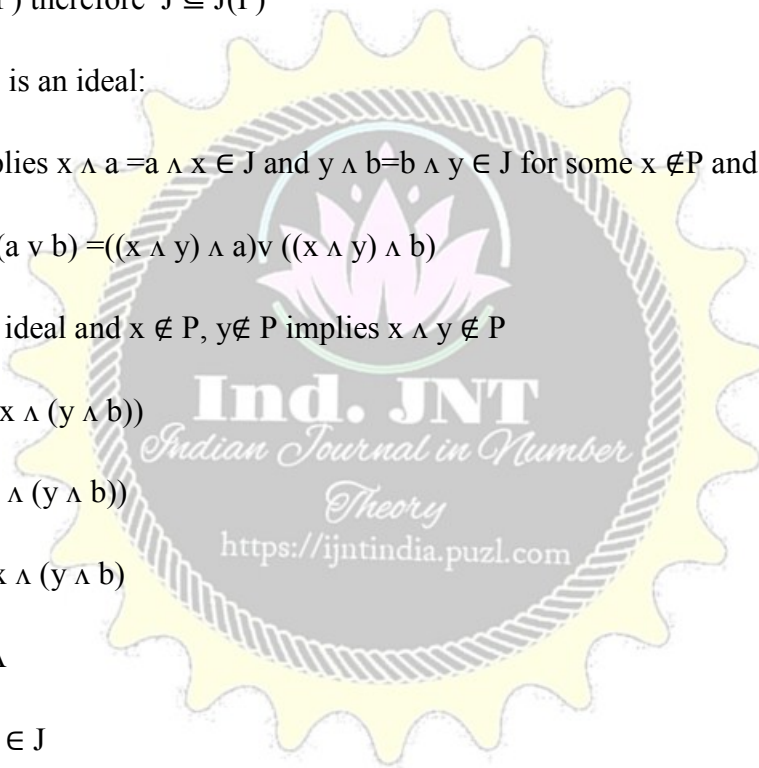
Implies $x \wedge (y \wedge b) \in J$

And as J is an ideal

Implies $((x \wedge a) \wedge y) \vee (x \wedge (y \wedge b)) \in J$

Implies $(x \wedge y) \wedge (a \vee b) \in J$ where $x \wedge y \notin P$

Implies $a \vee b \in J(P)$



Now let $a \in J(P)$ and $r \in A$

Then $x \wedge a \in J$ for some $x \notin P$

Now to show $r \wedge a \in J(P)$

Consider $x \wedge (r \wedge a) = x \wedge (a \wedge r)$

$= (x \wedge a) \wedge r$

As $x \wedge a \in J$ and $r \in A$ and J is an ideal so $(x \wedge a) \wedge r \in J$

Implies $x \wedge (r \wedge a) \in J$ and $x \notin P$

Implies $r \wedge a \in J(P)$

Therefore $J(P)$ is an ideal of A such that $J \subseteq J(P)$

Now let $a \in J(P)$

Then $x \wedge a \in J$ for some $x \notin P$

Implies $x \wedge a \in P$ since $J \subseteq P$

And since P is prime ideal and $x \notin P$

Implies $a \in P$

Implies $J(P) \subseteq P$

Theorem:III.1:

Let P be a prime ideal containing an ideal J of distributive q -lattice A and Q be a prime ideal such that $J \subseteq Q \subseteq P$ then $J(P) \subseteq Q$

Proof: We know $J(P) = \{ x \in A / x \wedge t \in J, \text{ for some } t \notin P \}$

$$= \{ x \in A / t \wedge x \in J, \text{ for some } t \notin P \}$$

By previous lemma $J \subseteq J(P) \subseteq P$

Let Q be a prime ideal such that $J \subseteq Q \subseteq P$



Now let $x \in J(P)$

Implies $t \wedge x \in J \subseteq Q$ for some $t \notin P$

Suppose $x \notin Q$

Since Q is prime, we get $t \in Q \subseteq P$

Which is contradiction to $t \notin P$

Thus $x \in Q$

Implies $J(P) \subseteq Q$

Defination:III.3:

Let A be distributive q - lattice. Two ideals I, J of A are called co-maximal

If $I \cup J = A$

Theorem:III.2:

Let A be distributive q -lattice then for any ideal J of A , then

Statements (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e)

(a) any two distinct minimal prime ideals belonging to J are co-maximal

(b) every prime ideal containing J , contains a unique minimal prime ideal belonging to J

(c) For each prime ideal P containing J ($J \subseteq P$) $J(p)$ is a prime ideal.

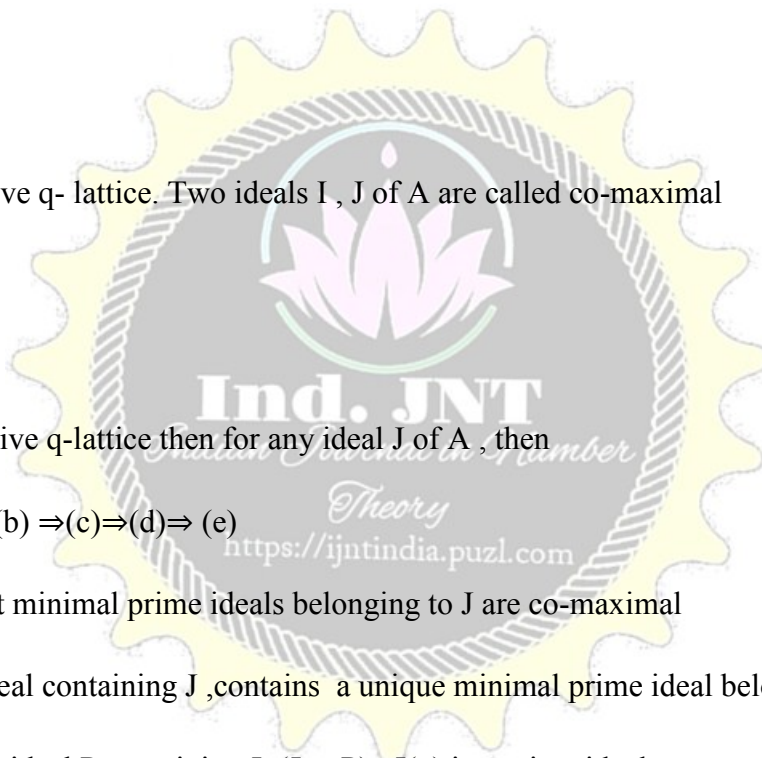
(d) for any $a, b \in A$, $a \wedge b \in J$ implies

$(a:J) \cup (b:J) = A$

(e) For any $a, b \in A$, $(a \vee b : J) \subseteq (a : J) \cup (b : J)$

Proof: (a) implies (b)

Let P be a prime ideal of A that is containing J ($J \subseteq P$) Suppose P contains two distinct minimal prime ideals Q_1 and Q_2 both belonging to J



Hence by a)

$$A = Q_1 \cup Q_2 \subseteq P$$

Which is a contradiction hence prime ideal P containing J , contains a unique minimal prime ideal belonging to J

(b) implies (c)

Assume that every prime ideal containing J (say $J \subseteq P$) contains a unique minimal prime ideal belonging to J

Let P be a prime ideal containing J , since $J(P) \subseteq P$ that is $J \subseteq J(P) \subseteq P$

We get that $J(P)$ is the unique minimal prime ideal belonging to J and contained in P therefore $J(P)$ will be a prime ideal.

(c) implies (d)

Assume for each prime ideal P containing J that is $J \subseteq P$, $J(P)$ is a prime ideal.

Let $a, b \in A$ such that $a \wedge b \in J$

Suppose $(a:J) \not\subseteq J$

Suppose $(a:J) \cup (b:J) \neq A$

Then there exists a prime ideal P containing J such that $(a:J) \cup (b:J) \subseteq P$

Hence $(a:J) \subseteq P$ and $(b:J) \subseteq P$

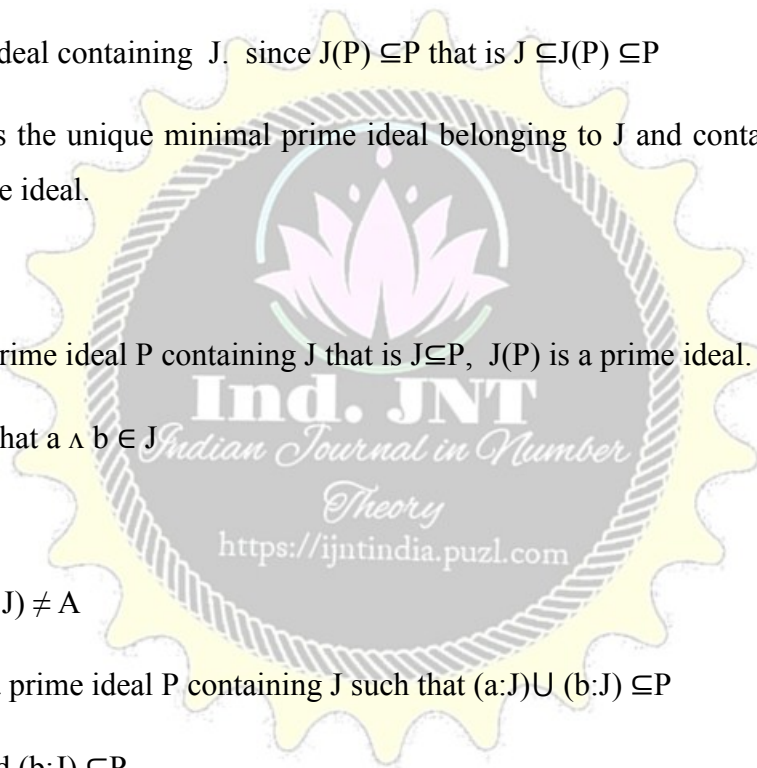
That is if $x \in (a:J)$ implies $x \in P$ and $x \in (b:J)$ implies $x \in P$ that is $x \wedge a \in J$ implies $x \in P$ and $x \wedge b \in J$ implies $x \in P$

Implies $a \notin J(P)$ and $b \notin J(P)$

Since $J(P)$ is prime ideal

We get $a \wedge b \notin J(P)$

Which is contradiction to that $a \wedge b \in J$ since $J \subseteq J(P)$



Hence $(a:J) \cup (b:J) = A$

(d) implies (e)

Assume for any $a, b \in A$, $a \wedge b \in J$ implies

$(a:J) \cup (b:J) = A$

Let $w \in (a \vee b : J)$ then $w \wedge (a \vee b) \in J$

That is $(w \wedge a) \vee (w \wedge b) \in J$ therefore from condition (d)

$(w \wedge a : J) \cup (w \wedge b : J) = A$

Now $w \in A$ implies $w \in (w \wedge a : J)$ or $w \in (w \wedge b : J)$

Implies $w \wedge (w \wedge a) \in J$ or $w \wedge (w \wedge b) \in J$

Implies $(w \wedge a) \in J$ or $(w \wedge b) \in J$

Implies $w \in (a : J)$ or $w \in (b : J)$

Implies $w \in (a : J) \cup (b : J)$

Implies $(a \vee b : J) \subseteq (a : J) \cup (b : J)$

Theorem:III.4:

Let A be distributive q -lattice. Define a relation Θ_J if and only if $(a : J) = (b : J)$ then Θ_J is a equivalence relation on A and J is congruence relation on A under condition $(a \vee c : J) \subseteq (a : J) \cap (c : J)$

Proof: i) As $(a:J) = (a:J)$ implies $(a, a) \in \Theta_J$ for all $a \in A$

Therefore Θ_J is reflexive

ii) If $(a, b) \in \Theta_J$ implies $(a:J) = (b:J)$ implies $(b:J) = (a:J)$

Implies $(b, a) \in \Theta_J$ therefore relation Θ_J is symmetric

iii) If $(a, b) \in \Theta_J$, $(b, c) \in \Theta_J$

Implies $(a:J) = (b:J)$, $(b:J) = (c:J)$

Implies $(a:J) = (b:J) = (c:J)$

Implies $(a:J) = (c:J)$

Implies $(a ,c) \in \Theta_J$ therefore relation Θ_J is transitive

Therefore relation Θ_J is an equivalence relation on A

Now let $(a, b) , (c ,d) \in \Theta_J$

Implies $(a:J) = (b:J)$ and $(c:J) = (d :J)$

Now, let $x \in (a \wedge c :J)$

$\Leftrightarrow x \wedge (a \wedge c) \in J$

$\Leftrightarrow (x \wedge a) \wedge c \in J$

$\Leftrightarrow x \wedge a \in (c : J)$

$\Leftrightarrow x \wedge a \in (d : J)$

$\Leftrightarrow (x \wedge a) \wedge d \in J$

$\Leftrightarrow x \wedge (a \wedge d) \in J$

$\Leftrightarrow x \wedge (d \wedge a) \in J$

$\Leftrightarrow (x \wedge d) \wedge a \in J$

$\Leftrightarrow x \wedge d \in (a : J)$

$\Leftrightarrow x \wedge d \in (b : J)$

$\Leftrightarrow (x \wedge d) \wedge b \in J$

$\Leftrightarrow x \wedge (d \wedge b) \in J$

$\Leftrightarrow x \wedge (b \wedge d) \in J$



$$\Leftrightarrow x \in (b \wedge d : J)$$

$$\text{Hence } (a \wedge c : J) = (b \wedge d : J)$$

$$\text{Therefore } (a \wedge c, b \wedge d) \in \Theta_J$$

$$\text{Now to show } (a \vee c, b \vee d) \in \Theta_J$$

$$\text{Let } x \in (a : J) \cap (c : J)$$

$$\text{Implies } x \in (a : J) \text{ and } x \in (c : J)$$

$$\text{Implies } x \wedge a \in J \text{ and } x \wedge c \in J \text{ and as } J \text{ is an ideal}$$

$$\text{Implies } (x \wedge a) \vee (x \wedge c) \in J$$

$$\text{Implies } x \wedge (a \vee c) \in J$$

$$\text{Implies } x \in (a \vee c : J)$$

$$\text{Hence } (a : J) \cap (c : J) \subseteq (a \vee c : J) \text{ and from given condition}$$

$$(a \vee c : J) \subseteq (a : J) \cap (c : J)$$

$$\text{Therefore } (a \vee c : J) = (a : J) \cap (c : J)$$

$$\text{Implies } (a \vee c : J) = (b : J) \cap (d : J)$$

$$\text{Implies } (a \vee c : J) = (b \vee d : J)$$

$$\text{Hence } (a \vee c, b \vee d) \in \Theta_J$$

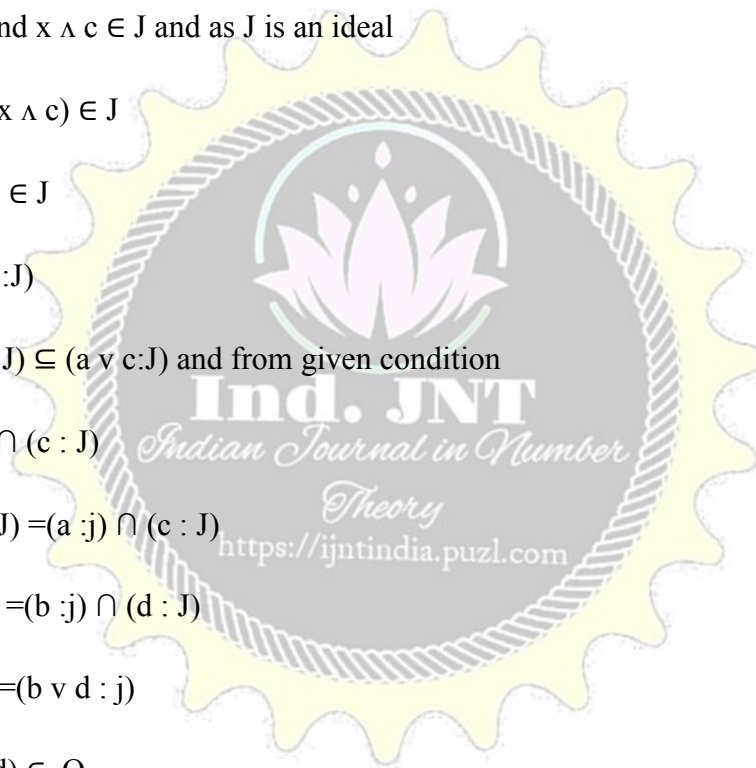
Theorem:III.5:

Let A_1 and A_2 be two distributive q-lattices and $f: A_1 \rightarrow A_2$ be a homomorphism then

1) If J is an ideal of A_2 then $f^{-1}(J) = \{ x \in A_1 / f(x) \in J \subseteq A_2 \}$ is an ideal of A_1

2) If J is filter of A_2 then $f^{-1}(J) = \{ x \in A_1 / f(x) \in J \subseteq A_2 \}$ is filter of A_1

Proof: Let $f: A_1 \rightarrow A_2$ be a homomorphism



1) Let J is an ideal of A_2 and $f^{-1} = \{ x \in A_1 / f(x) \in J \subseteq A_2 \}$

Let $x, y \in f^{-1}(J)$ then $f(x), f(y) \in J$

As J is an ideal, $f(x) \vee f(y) = f(x \vee y) \in J$

Therefore $x \vee y \in f^{-1}(J)$

Now if $x \in f^{-1}(J)$ and $r \in A_1$

Then $f(x) \in J, f(r) \in A_2$ and as J is an ideal of A_2 , therefore

$f(x) \wedge f(r) = f(x \wedge r) \in J$

Thus $x \wedge r \in f^{-1}(J)$ therefore $f^{-1}(J) = \{ x \in A_1 / f(x) \in J \subseteq A_2 \}$ is an ideal of A_1

2) Let J is filter of A_2 and $f^{-1}(J) = \{ x \in A_1 / f(x) \in J \subseteq A_2 \}$

Let $x, y \in f^{-1}(J)$ then $f(x), f(y) \in J$

As J is filter, $f(x) \wedge f(y) = f(x \wedge y) \in J$

Therefore $x \wedge y \in f^{-1}(J)$

Now if $x \in f^{-1}(J)$ and $r \in A_1$

Then $f(x) \in J, f(r) \in A_2$ and as J is filter of A_2 , therefore

$f(x) \vee f(r) = f(x \vee r) \in J$

Thus $x \vee r \in f^{-1}(J)$ therefore $f^{-1}(J) = \{ x \in A_1 / f(x) \in J \subseteq A_2 \}$ is filter of A_1

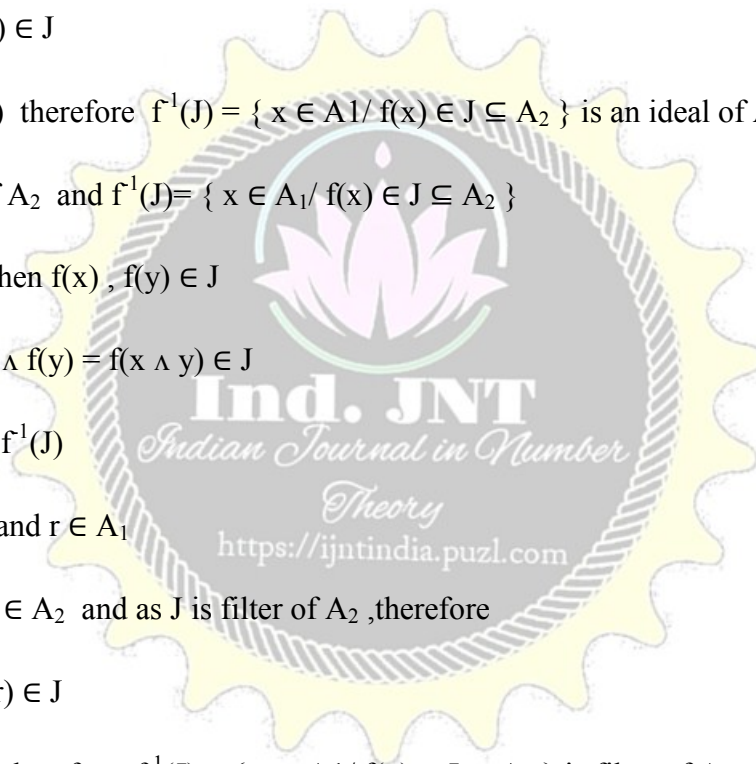
Theorem: III.6:

Let A_1 and A_2 be two distributive q-lattices and $f: A_1 \rightarrow A_2$ be a homomorphism then

1) $f(P^*) \subseteq (f(P))^*$

Proof: let $x \in f(P:I)$

Implies there exists $y \in (P:I)$ such that $f(y) = x \in f(P:I)$



Implies $y \wedge a \in I$ for all $a \in P$

Implies $f(y \wedge a) \in f(I)$ for all $a \in P$

Implies $f(y) \wedge f(a) \in f(I)$ for all $a \in P$ that is for all $f(a) \in f(P)$

Implies $f(y) \in \{ f(P) : f(I) \}$

Implies $x \in \{ f(P) : f(I) \}$

Implies $f(P:I) \subseteq \{ f(P) : f(I) \}$

Therefore $f(P^*) \subseteq (f(P))^*$



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