Ind. JNT
(Indian Journal in Number theory)

Received: *13.10.2015* ***Year:*** *2017*, ***Pages:*** *47-53*
Published: *30.11.2017* *Original Article***

A SUBCLASS OF ANALYTIC AND MULTIVALENT FUNCTIONS GENERALIZED BY POSITIVE COEFFICIENT USING FRACTIONAL CALCULUS OPERATORS FROM GEOMETRIC FUNCTION THEORY

Phulambrikar Arunkumar Prabhakarrao
Department of Mathematics
J. J. T. University, Zhunzhunu, Rajasthan

Abstract- $f(z)$Is the class of function in the form of

$$f(z) = Z^p - \sum_{k=n+p}^{\infty} a_k z^k (a_k \geq 0, p, n \in IN)$$

We introduce the subclass $k_\mu^{\lambda,\phi,n}(n,p,\propto)$ by using fractional calculus operator connection to the well known and some new properties be discussed. Also for this subclass necessary and sufficient condition for a function to be in $k_\mu^{\lambda,\phi,n}(n,p,\propto)$ is obtained. We have also obtained the radius of star likeness, convexity, close to convexity, distortion, fractional calculus and inclusion property for functions in the subclass. Furthermore, we give an application of class preserving integral operator.

INTRODUCTION:

Let M (n, p) denotes $f(z)$ functions of the form,

$$f(z) = Z^p - \sum_{k=n+p}^{\infty} a_k z^k (a_k \geq 0, p, n \in IN) \quad \ldots\ldots\ldots\ldots\ldots.(1.1)$$

Have open disc $U = \{z : z \in c \,\&|z| < 1\}$ $\;also\;$ $U = \{z : z \in c \,\&\, 0 < |z| < 1\}$

$$\left| \frac{zJ_0^{1+\lambda,1+\phi,1+n}\{f(z)\} + \mu z^2 J_{0,z}^{2+\lambda+\phi,2+n}\{f(z)\}}{(1-\mu)J_{0,z}^{\lambda,\phi,n}\{f(z)\} + \mu z J_0^{1+\lambda,1+\phi,1+n}\{f(x)\}} - (p-\phi) \right| < \propto \quad \ldots\ldots\ldots\ldots\ldots(1.2)$$

Where, $J_{0,z}^{\lambda,\phi,n}$ denotes an operators of fractional calculus.

Definition 1: The function $f(z)$of fractional derivatives of order λ is

$$D_z^\lambda f(z) = \frac{1}{/(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)\lambda} dt (0 \leq \lambda \leq 1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.3)$$

With multiplicity of remove by $(z-t)^{-\lambda-1}$

Definition 2: The generalized fractional integral operators $I_0^{\lambda,\beta,n}$ of a function $f(z)$is,

$$I_0^{\lambda,\beta,n}\{f(z)\} = \frac{z^{-\lambda-\beta}}{\Gamma(1-\lambda)}\int_0^z (z-t)^{\lambda-1} f(t)_2 f,\left(\lambda+\beta-n;\lambda;1-\frac{t}{2}\right)dt \qquad \ldots..(1.4)$$

Where the function $f(z)$is analytic is simply connected region of z-plane containing the origin.

$$f(z) = 0(|Z|^{\epsilon}); z \to 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.5)$$

$$\epsilon > \max\{0,\beta-n\}-1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(1.6)$$

By requiring $\log(z-t)$to real when (z-t)>0.

By the definition above we note that

$$Q^{\lambda,\phi,n\,(n;p;\propto)} = k_0^{\lambda,\phi,n}(n;p;\propto)(n,p\in IN; 0\le\lambda\le 1; 0\le\propto\le p) \qquad \ldots\ldots\ldots\ldots\ldots(i)$$

$$\Omega_\lambda(n;p;\propto) = Q^{\lambda,\lambda,n}(n;p;\propto)(n,p\in IN; 0\le\lambda\le 1; 0\le\propto\le p) \qquad \ldots\ldots\ldots\ldots\ldots(ii)$$

$$\Delta_\lambda(n;p;\propto) = R^{\lambda,\lambda,n}(n;p;\propto)(n,p\in IN; 0\le\lambda\le 1; 0\le\propto\le p) \qquad \ldots\ldots\ldots\ldots\ldots(iii)$$

$$R^{\lambda,\phi,n\,(n;p;\propto)} = k_1^{\lambda,\phi,n}(n;p;\propto)(n,p\in IN; 0\le\lambda\le 1; 0\le\propto\le p) \qquad \ldots\ldots\ldots\ldots\ldots(iv)$$

$$S(p;\propto) = S_1(p;\propto)(p\in IN; 0\le\propto\le p) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(v)$$

$$C(p;\propto) = C_1(p;\propto)(p\in IN; 0\le\propto\le p) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(vi)$$

$$C_n(p;\propto) = \Delta_0(n;p;\propto)(n,p\in IN; 0\le\propto\le p) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(vii)$$

$$S_n(p;\propto) = \Omega_0(n;p;\propto)(n,p\in IN; 0\le\propto\le p) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(viii)$$

The classes $S_n(p;\propto)$ and $S^*(p;\propto)$ consist of star like function
$(p-x);(0\le\propto\le p)$P-valiantly convex function of order$(p-x);(0\le\propto\le p)$
$C_n(p;\propto)\&C^*(p,\propto)$ Also star like function of order $(1-x)0\le\propto\le 1$ and $C^*(\propto) = C^*(1;\propto)$ is the class of convex function of order
$(1-\propto);(0<\propto\le 1)$.

2. Coefficient Bounds:-

As $f(z)\in M(n,p)$ of class $C_n(p,\propto)$ if and only if,

$$\sum_{k=n+p}^{\infty} k(k-p+\propto)a_k \le \propto p \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.1)$$

$(n,p\in IN; 0\le\propto\le p)$

When we take n=1, p=1 above and

$$\sum_{k=n+p}^{\infty} k(k-p+\propto)a_k \le \propto$$

Get star like and convex functions.

Lemma1: n is a positive integer of $<0\le\beta<1$ $and\ r,n\in IR$ such that $\gamma<1+p; n>\max\{\beta,\gamma\}-(1-\gamma)$.

$$n \geq \frac{\gamma(n-\beta)}{\beta} - (1+p) \qquad \ldots\ldots\ldots\ldots\ldots\ldots..(2.2)$$

Where $f(z) \in k_{\mu}^{\lambda,\phi,n}|(n;\gamma;\propto)$ then

$$\left||J_0^{\beta,\gamma,n} f(z)\right| - \frac{\Gamma(1+p)\Gamma(1+p+n-\gamma)}{\Gamma(1+p-r)\Gamma(1+p+n-\beta)}|z|^{p-\gamma}| \leq$$

$$\frac{\{\propto \Gamma(1+p)\Gamma(1+n+p+n-\gamma)\Gamma(1+p+n-\phi)[1+\mu(p-\phi-1)]\Gamma(1+n+p-\phi)\Gamma(1+n+p+n-\lambda)|z|^{1+p-\gamma}\}}{\{\Gamma(1+n+p-\gamma)\Gamma(1+n+p+n-\gamma)\Gamma(1+p-\phi)\Gamma(1+p+n-\lambda)(n+\propto)[1+\mu(n+p-\phi-1)]\Gamma(1+n+p+n-\phi)\}} \ldots\ldots$$

$$\ldots\ldots(2.3)$$

Of $z \in \cup$ if $\gamma \leq p$ & $z \in \cup^*$ *if* $\gamma > p$ the result is sharp for the above function.

Proof:-

$$J_{0,z}^{\beta,r,n} f(z) = \frac{\Gamma(1+p)\Gamma(1+p+n-\gamma)}{\Gamma(1+p-\gamma)\Gamma(1+p+n-\beta)} Z^{p-\gamma} - \sum_{k=n+p}^{\infty} \frac{\Gamma(1+k)\Gamma(1+k+n-\gamma)}{\Gamma(1+k-\gamma)\Gamma(1+k+n-\beta)} a_k Z^{k-\gamma} \qquad \ldots\ldots\ldots.(2.4)$$

$$\leq \{\propto \Gamma(1+n+p-\gamma+n)\Gamma(1+p)\Gamma(1+p+n-\phi)[1+\mu(p-\phi-1)]\Gamma(1+n+p-\phi)\Gamma(1+n+p+n-\lambda)|z|^{1+p-\gamma}\}$$

$$\{\Gamma(1+n+p-\gamma)\Gamma(1+n+p+n-\beta)\Gamma(1+p-\phi)\Gamma(1+p+n-\lambda)(n+\propto)[1+\mu(n+p-\phi-1)]\Gamma(1+n+p+n-\phi)\}$$

$$\leq \frac{\{\propto \Gamma(1+n+p-\gamma+n)\Gamma(1+p)\Gamma(1+p+n-\phi)[1+\mu(p-\phi-1)]\Gamma(1+n+p-\phi)\Gamma(1+n+p+n-\lambda)|z|^{1+p-\gamma}\}}{\{\Gamma(1+n+p-\gamma)\Gamma(1+n+p+n-\beta)\Gamma(1+p-\phi)\Gamma(1+p+n-\lambda)(n+\propto)[1+\mu(n+p-\phi-1)]\Gamma(1+n+p+n-\phi)\}}$$

$$\ldots\ldots(2.5)$$

$$h(k) = \frac{\Gamma(1+k)\Gamma(1+k+n-\gamma)}{\Gamma(1+k-\gamma)\Gamma(1+k+\beta+n)}$$

$(k \geq n+p; n; p \in IN)$

Where $h(k)$is a non increasing function of k$(k \geq n+p)$

$$0 \leq h(k) \leq h(n+p) = \frac{\Gamma(1+n+p)\Gamma(1+n+p+n-\gamma)}{\Gamma(1+n+p-\gamma)\Gamma(1+n+p+\beta+n)}$$

$(n,p \in IN)$ $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2.6)$

By (2.5) with (2.4) & (2.6) we get $n \geq \frac{\gamma(n-\beta)}{\beta} - 1 + p$& (2.3) here Proved.

Corollary 1:- If $f(z) \in k_{\mu}^{\lambda,\phi,n(n;p;\propto)}$ then

$$\left|\left|D_z^{-\beta} f(z)\right| - \frac{\Gamma(1+p)}{\Gamma(1+p+\beta)}|z|^{p+\beta}\right| \leq$$

$$\frac{\{\propto \Gamma(1+p)\Gamma(1+p+n-\phi)[1+\mu(p-\phi-1)]\Gamma(1+n+p-\phi)\Gamma(1+n+p+n-\lambda)|z|^{1+p+\beta}\}}{\{\Gamma(1+n+\beta+p)\Gamma(1+p-\phi)\Gamma(1+p+n-\lambda)(n+\propto)[1+\mu(n+p-\phi-1)]\Gamma(1+n+p+n-\phi)\}}$$

... ... (2.7)

Where $\beta(\beta>0), z \in \cup$ and $n, p \in IN$

Corollary 2: If $f(z) \in k_{\mu}^{\lambda,\phi,n}(n;p;\propto)$ then

$$\frac{||f(z)-|z|^{p}| \leq \{\propto \Gamma(1+p)\Gamma(1+p+n-\phi)[1+\mu(p-\phi-1)]\Gamma(1+n+p+n-\lambda)|z|^{p+1}\}|}{\{[(1+n+p)]\Gamma(1+p-\phi)\Gamma(1+p+n-\lambda)(n+\propto)[1+\mu(n+p-\phi-1)]\Gamma(1+n+p+n-\phi)\}}$$

... (2.8)

3. PROPERTIES:

Further we find the star likeness of the functions, convexity, close to convex function.

Lemma 2: If

$$f_j(z) = Z^p - \sum_{k=n+p}^{\infty} a_{k,j} Z^k$$

$\left(a_{k_j} \geq 0, p, n \in IN, \ni = 1,2, \dots\dots l\right)$ is $K_{\mu}^{\lambda,\phi,n}(n,p,\propto)$ of

$$h(z) = \frac{1}{l} - \sum_{j=1}^{\infty} f_j^{(z)} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (2.9)$$

Also belongs to $K_{\mu}^{\lambda,\phi,n}(n,p,\propto)$

Proof: by the definition of $h(z)$ we have,

$$h(z) = z^p - \sum_{k=n+p}^{\infty} \left(\frac{1}{l}\sum_{i=1}^{l} a_{k_j}\right) z^h \quad \dots\dots\dots\dots\dots\dots\dots (3.1)$$

$f_j(z) \in k_{\mu}^{\lambda,\phi,n}(n,p,\propto)$ $\quad\quad (j = 1,2, \dots\dots l)$

By,

$$\sum_{k=n+p}^{\infty} \frac{(k-p+\propto)[1+\mu(k-\phi-1)]\Gamma(1+k)\Gamma(1+k+\eta-\phi)}{\Gamma(1+k-\phi)\Gamma(1+k+\eta-\lambda)} a_{k,j}$$

$$\leq \frac{\propto \Gamma(1+p)\Gamma(1+p+\eta+\phi)[1+\mu(p-\phi-1)]}{\Gamma(1+p-\phi)\Gamma(1+p+\eta-\lambda)} \quad \dots\dots. (3.2)$$

Where $(n, p, \in IN; 0 < \propto \leq p, 0 \leq \mu \leq 1; 0 \leq \lambda \leq 1; \phi, \eta \in IR; \phi \in p; n > \max\{\lambda, \phi\} - (1+p))$

By the simple method $h(z)$ deduced in the $k_{\mu}^{\lambda,\phi,\eta}(n,p,\alpha)$

Lemma 3:

If $f(z) \in k_{\mu}^{\lambda,\phi,\eta}(n,p,\alpha)$ then close-to-convex function of order d in $|z| < r_3 \; as,$

$$r_3 = inf_k \left\{\frac{p-d}{k} g(k,n,p,\mu,\propto,\lambda,\phi,\eta)\right\}^{\frac{1}{k}-p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.3)$$

Proof: As $0 \le d < p$ of $f(z) \in k_{\mu}^{\lambda,\phi,n}(n,p,\alpha)$equivalent to,

$$\left|\frac{f^{|}(z)}{z^{p-1}} - p\right| < p - d \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (3.4)$$

By simplifying the required result for $g(k,n,p,\mu,\alpha,\lambda,\phi,\eta)$r in

$$g(k,n,p,\mu,\alpha,\lambda,\phi,\eta) = \frac{(k-p+\propto)[1+\mu(k-\phi-1)]\Gamma(1+k)}{\Gamma(1+k-\phi)\Gamma(1+k+\eta-\lambda)\propto} * \frac{\Gamma(1+k+n-\phi)\Gamma(1+p-\phi)\Gamma(1+p+n-\lambda)}{\Gamma(1+p)\Gamma(1+p+\eta-\phi)[1+\mu(p-\phi-1)]} \quad \ldots\ldots (3.5)$$

4. Integral operator of $k_{\mu}^{\lambda,\phi,n}(n,p,\alpha)$.

$f(z)$Is defined as,

$$H(z) = P_{c,p}^{d} f(z) = \frac{(c+p)^d}{\Gamma(d) z^c} \int_0^z t^{c-1} (log\frac{z}{t})^{d-1} f(t).dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.6)$$

$(d > 0, c > -p; z \in \cup)$

Is komatu integral operator [5].

Lemma 4: If then the H (z) is given by (3.6) is p-valent in $|z| \le r_4$ when,

$$r_4 = inf_k \left\{\frac{ph(k)(k-p+\propto)}{k \propto h(p)}\right\}^{\frac{1}{k}-p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.7)$$

Proof: - As,

$$H(z) = z^p - \sum_{k=p+n}^{\infty} \left(\frac{c+p}{c+k}\right)^d a_k z^k \quad \ldots\ldots\ldots\ldots\ldots (3.8)$$

To prove, $\left|\frac{H^{|}(z)}{z^{p-1}} - p\right| \le p \;\; in \, |z| < r_4$ ……………………………….(3.9)

Where r_4 is gives result (3.7) with (3.8) we have,

$$\left|\frac{H`(z)}{z^{p-1}}-p\right| = |-\sum_{k=p+n}^{\infty}\left(\frac{c+p}{c+k}\right)^{d} a_k z^k| \leq \sum_{k=p+n}^{\infty}\left(\frac{c+p}{c+k}\right)^{d} a_k|z|^{k-p} \quad \ldots\ldots\ldots(3.10)$$

The above inequality bounded by p if

$$\sum_{k=p+n}^{\infty}\left(\frac{c+p}{c+k}\right)^{d} a_k|z|^{k-p} \leq 1 \quad \ldots\ldots\ldots(3.11)$$

But $H(z) \in k_{\mu}^{\lambda,\phi,\eta}(n,p,\alpha)$ and function $f(z) \in M(n,p)$of .

$$\sum_{k=n+p}^{\infty}\frac{(k+p)[1+\mu(k-\phi-1)\Gamma(1+k)\Gamma(1+\lambda,n-\phi)]}{\Gamma(1+k-\phi)\Gamma(1+p+\eta-\lambda)}a_k \leq \frac{\propto \Gamma(1+p)\Gamma(1+p+n-\phi)[1+\mu(p-\phi-1)]}{\Gamma(1+p-\phi)\Gamma(1+p+\eta-\lambda)} \quad \ldots\ldots\ldots(3.12)$$

$$\sum_{k=n+p}^{\infty}\frac{(k-p+\propto)s(k)\left(\frac{c+p}{c+k}\right)^{d}}{\propto s(p)}a_k \leq 1$$

And form (3.11) and (3.10) will hold of

$$|z| \leq \frac{ps(k)(k-p+\propto)^{\frac{1}{k}-p}}{k \propto s(p)} \quad for\ k \geq p+n, n \in IN \quad \ldots\ldots\ldots\ldots\ldots(3.13)$$

The above lemma leads to important assertion of the $|z| = r_4$.

5. Extreme Points:

The extreme points of the class $k_{\mu}^{\lambda,\phi,\eta}(n,p,\alpha)$ are,

$f_p(z) = z^p$and$f_\lambda(z) = z^p = \frac{\propto s(p)}{(k-p+\propto)s(k)}z^k; k \geq p+1.$

Lemma 5: Let $f_p(z) = z^p \& f_k(z) = z^p - \frac{\propto s(p)}{(k-p+\propto)s_{(k)}}z^k$

$$f(z) = \sum_{k=p+n}^{\infty}\lambda_k f_k(z) \qquad (n \in IN_0) \quad \ldots\ldots\ldots\ldots\ldots(3.14)$$

$$\sum_{k=p+n}^{\infty}\lambda_k = 1$$

Proof:- As $f(z)$ is expressible in the form

$$f(z) = \sum_{k=p+n}^{\infty}\lambda_k f_k(z) \qquad (n \in IN_0)$$

$$= z^p - \sum_{k=p+n}^{\infty} \frac{\propto s(p)}{(k-p+\propto)s(k)} \lambda_k z^k \qquad (n \in IN)$$

Now,

$$\sum_{k=p+n}^{\infty} \frac{\propto s(p)}{(k-p+\propto)s(k)} \frac{(k-p+\propto)s(k)}{\propto s(p)}$$

$$= \sum_{k=p+n}^{\infty} \lambda_k = 1 - \lambda_p \leq 1 \quad (n \in IN) \qquad \ldots\ldots\ldots\ldots (3.15)$$

So, $f(z) \in k_\mu^{\lambda,\phi,n}(n,p,\alpha)$

As $f(z) \in k_\mu^{\lambda,\phi,n}(n,p,\alpha)$ then, setting

$\lambda_k = \left(\frac{\propto s(p)}{(k-p+\propto)s(k)} a_k\right)$ And $\lambda_p = 1 - \sum_{k=p+n}^{\infty} \lambda_k (n \in IN)$ …………… (3.16)

Therefore we expressed as function $f(z)$ is in the form of (3.14).

References:

1. S. Kanas, Lenear operator associated with k- uniformly convex function, integral transform. Spec. function vol 9(2) (2000) 125-130.
2. H. M. Srivastava, Yi Ling, G, Bao "Some Distortion inequalities associated with the fractional derivatives of analytic and univalent function IPAM (2002) (1-9).
3. R. K. Raina, J.H. Choi some result connected with a subclass of analytic function involving certain fractional calculus operator JFG (2003) (20-24).
4. A subclass of uniformly convex function associated with certain fractional calculus operators JIPAM (2005) (2-6).
5. N. Magesh, G. Murugusundaramoorthy "An application of second order differential inequalities based on linear and integral operators, IJMSEA (2008) (108-113).
6. S. K. Lee, S.M. Khairnar, Meena More "some application & Properties of GFC operators of U-V & M.V.F. KJM (2009) (129-145).

****