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SOME ALTERNATIVE PROOF OF RIEMANN INTEGRAL PROPERTY

P. G. Andhare

Department of Mathematics, R. B. N. B, College,
Srirampur, Dist. Ahmednagar. Pin 413709

Abstract- In the present paper we give alternative proof of Riemann integral property [2, P.166], if f ϵ R [a, b] and λ is any real number, then λ f ϵ R[a, b] and $\int_a^b \lambda f = \lambda \int_a^b f$.

INTRODUCTION:

We give some definitions.

1.1 Definition: Let I_k be any bounded interval of real numbers. Let f be a bounded real valued function on I_k.We define M [f;I_k] and m[f; I_k] as

$$M[f;I_k]=l.u.b.\,of\ f(x)$$

$$x \in I_k$$

and $$m[f;I_k]=glb\ of\ f(x)$$

$$x \in I_k$$

1.2 Definition: Let f be bounded function on closed bounded interval [a,b]. let σ be subdivision of closed interval [a, b].We define $U[f;\sigma]$ called upper sum for f corresponding to sub division σ as

$$U[f;\sigma] = \sum_{k=1}^{n} M\,[f;I_k]\ |I_k|$$

where $I_1, I_2, I_3, \ldots\ldots I_n$ are component interval of subdivision σ . Similarly , the lower sum $L[f;\sigma]$ is de fined as

$$L[f;\sigma] = \sum_{k=1}^{n} m\,[f; I_k]\ |I_k|$$

1.3 Definition : Let f be a bounded function on closed bounded interval [a, b *]*. We define $\bar{\int_a^b}$ f (x) dx called the upper integral of over [a, b],as $\bar{\int_a^b}$ f (x) $= g.l.b.U[f;\sigma]$

where g.l.b. is taken over all subdivisions σ of [a, b]. Similarly , we define $\underline{\int_a^b}$ f (x) dx called the lower integral of over [a, b],as $\underline{\int_a^b}$ f (x) $= l.u,b.L[f;\sigma]$

where l. u .b. is taken over all subdivisions σ of [a, b].

1.4 Existence of the Riemann Integral: Let f be a bounded function on closed bounded interval [a, b *]*.Then f ϵ R $[a,b]$ if and only if f is continuous at almost every point in [a ,b].

MAIN RESULT:

We give alternative proof of following theorem

2.1Theorem : If f ϵ R $[a,b]$ and λ is any real number, then λ f ϵ R$[a,b]$ and $\int_a^b \lambda f = \lambda \int_a^b f$

Proof :

Since f ϵ R $[a,b]$ ∴ By existence theorem on Riemann integration f is continuous at almost every point in [a, b] . But we know for any real λ , λ f is continuous where f is continuous .Therefore λf is continuous at almost every point in [a, b] . Therefore by existence theorem on Riemann integration

λ f ϵ R$[a,b]$.

Case (I) : If λ=0 ,then $\int_a^b \lambda f = \int_a^b 0f = 0$ ……………………….(1)

Also $\lambda \int_a^b f = 0 \int_a^b f = 0$ ……………………….(2)

From (1) and (2) $\int_a^b \lambda f = \lambda \int_a^b f$

Case (II) : Suppose λ > 0 . Let σ be subdivision of closed interval [a ,b].Let $I_1, I_2, I_3, \ldots\ldots I_n$ be component interval of subdivision σ .

By definition $m[f; I_k] = glb\ of\ f(x)$
x ϵ I_k

$m[\lambda f; I_k] = glb\ of\ \lambda f(x)$
x ϵ I_k

$$= \lambda\, glb\ of\ f(x)$$

$$x \in I_k$$

$$m[\lambda f; I_k] = \lambda\, m[f; I_k]$$

$$\sum_{k=1}^{n} m[\lambda f; I_k]|I_k| = \lambda \sum_{k=1}^{n} m[f; I_k]\ |I_k|$$

$$L[\lambda f; \sigma] = \lambda\ L[f; \sigma]$$

$$l.u.b.\ L[\lambda f; \sigma] = \lambda\ \text{l.u.b.}\ L[f; \sigma]$$

$$\sigma \qquad\qquad \sigma$$

$$\underline{\int_a^b} \lambda f = \lambda \underline{\int_a^b} f$$

But $f \in R[a, b]$ $\therefore$ $\overline{\int_a^b} f = \underline{\int_a^b} f = \int_a^b f$

$\lambda f \in R[a, b]$ $\therefore$ $\overline{\int_a^b} \lambda f = \underline{\int_a^b} \lambda f = \int_a^b \lambda f$

$$\therefore \int_a^b \lambda f = \lambda \int_a^b f$$

Case (III) : Suppose $\lambda < 0$.

Clearly $g.l.b.\ of\ \{-f(x)\} = -l.u.b.\{f(x)\}$

$$x \in I_k \qquad\qquad x \in I_k$$

$$m[-f; I_k]|I_k| = -\ M[f; I_k]|I_k|$$

$$\sum_{k=1}^{n} m[-f; I_k]|I_k| = -\sum_{k=1}^{n} M\,[f; I_k]\ |I_k|$$

$$L[-f; \sigma] = -\ U[f; \sigma]$$

$$\underset{x\,\epsilon\, I_k}{l.u.b.}\, L[-f;\sigma] = \underset{x\,\epsilon\, I_k}{l.u.b.}\{-U[f;\sigma]\}$$

$$l.u.b.\, L[-f;\sigma] = \{-g.l.b.\, U[f;\sigma]\}$$

$$\underline{\int}_a^b f = -\overline{\int}_a^b f$$

Since f ϵ R$[a,b]$ and -f ϵ R$[a,b]$

$$\therefore \overline{\int}_a^b f = \underline{\int}_a^b f = \int_a^b f \text{ and } \overline{\int}_a^b -f = \underline{\int}_a^b -f = \int_a^b -f$$

$$\therefore \int_a^b -f = -\int_a^b f \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Since $\lambda < 0$ Put $\lambda = -\mu$, then $\mu > 0$

$$\therefore \int_a^b \lambda f = \int_a^b -\mu\, f$$

$$= -\int_a^b \mu\, f \qquad \text{by using (3)}$$

$$= -\mu \int_a^b f \qquad \text{by case (II)}$$

$$= \lambda \int_a^b f$$

Thus for any real number $\lambda \,\epsilon\, R$

$$\int_a^b \lambda f = \lambda \int_a^b f$$

References:

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[2] R. R. Goldberg: Methods of real Analysis; Oxford & IBH Publishing Co. (1970)