

SOME ALTERNATIVE PROOF OF RIEMANN INTEGRAL PROPERTY

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Abstract- In the present paper we give alternative proof of Riemann integral property [2, P.166], if $f \in R[a, b]$ and λ is any real number, then $\lambda f \in R[a, b]$ and $\int_a^b \lambda f = \lambda \int_a^b f$.

INTRODUCTION:

We give some definitions.

1.1 Definition: Let I_k be any bounded interval of real numbers. Let f be a bounded real valued function on I_k . We define $M[f; I_k]$ and $m[f; I_k]$ as

$$M[f; I_k] = \text{l.u.b. of } f(x)$$

$$x \in I_k$$

$$\text{and } m[f; I_k] = \text{g.l.b. of } f(x)$$

$$x \in I_k$$

1.2 Definition: Let f be bounded function on closed bounded interval $[a, b]$. let σ be subdivision of closed interval $[a, b]$. We define $U[f; \sigma]$ called upper sum for f corresponding to sub division σ as

$$U[f; \sigma] = \sum_{k=1}^n M[f; I_k] |I_k|$$

where $I_1, I_2, I_3, \dots, I_n$ are component interval of subdivision σ . Similarly, the lower sum $L[f; \sigma]$ is defined as

$$L[f; \sigma] = \sum_{k=1}^n m[f; I_k] |I_k|$$

1.3 Definition : Let f be a bounded function on closed bounded interval $[a, b]$. We define $\int_a^b f(x) dx$ called the upper integral of over $[a, b]$, as $\int_a^b f(x) = g.l.b. U[f; \sigma]$

where $g.l.b.$ is taken over all subdivisions σ of $[a, b]$. Similarly, we define $\int_a^b f(x) dx$ called the lower integral of over $[a, b]$, as $\int_a^b f(x) = l.u.b. L[f; \sigma]$

where $l.u.b.$ is taken over all subdivisions σ of $[a, b]$.

1.4 Existence of the Riemann Integral: Let f be a bounded function on closed bounded interval $[a, b]$. Then $f \in R[a, b]$ if and only if f is continuous at almost every point in $[a, b]$.

MAIN RESULT:

We give alternative proof of following theorem

2.1 Theorem : If $f \in R[a, b]$ and λ is any real number, then $\lambda f \in R[a, b]$ and $\int_a^b \lambda f = \lambda \int_a^b f$

Proof :

Since $f \in R[a, b] \therefore$ By existence theorem on Riemann integration f is continuous at almost every point in $[a, b]$. But we know for any real λ , λf is continuous where f is continuous. Therefore λf is continuous at almost every point in $[a, b]$. Therefore by existence theorem on Riemann integration

$\lambda f \in R[a, b]$.

Case (I) : If $\lambda=0$, then $\int_a^b \lambda f = \int_a^b 0f = 0 \dots\dots\dots(1)$

Also $\lambda \int_a^b f = 0 \int_a^b f = 0 \dots\dots\dots(2)$

From (1) and (2) $\int_a^b \lambda f = \lambda \int_a^b f$

Case (II) : Suppose $\lambda > 0$. Let σ be subdivision of closed interval $[a, b]$. Let $I_1, I_2, I_3, \dots, I_n$ be component interval of subdivision σ .

By definition $m[f; I_k] = glb \text{ of } f(x)$

$$x \in I_k$$

$m[\lambda f; I_k] = glb \text{ of } \lambda f(x)$

$$x \in I_k$$

$$= \lambda \text{ glb of } f(x)$$

$$x \in I_k$$

$$m[\lambda f; I_k] = \lambda m[f; I_k]$$

$$\sum_{k=1}^n m[\lambda f; I_k] |I_k| = \lambda \sum_{k=1}^n m[f; I_k] |I_k|$$

$$L[\lambda f; \sigma] = \lambda L[f; \sigma]$$

$$l. u. b. L[\lambda f; \sigma] = \lambda l. u. b. L[f; \sigma]$$

$$\int_a^b \lambda f = \lambda \int_a^b f$$

But $f \in R[a, b] \therefore \int_a^b f = \int_a^b f = \int_a^b f$

$\lambda f \in R[a, b] \therefore \int_a^b \lambda f = \int_a^b \lambda f = \int_a^b \lambda f$

$$\therefore \int_a^b \lambda f = \lambda \int_a^b f$$

Case (III) : Suppose $\lambda < 0$.

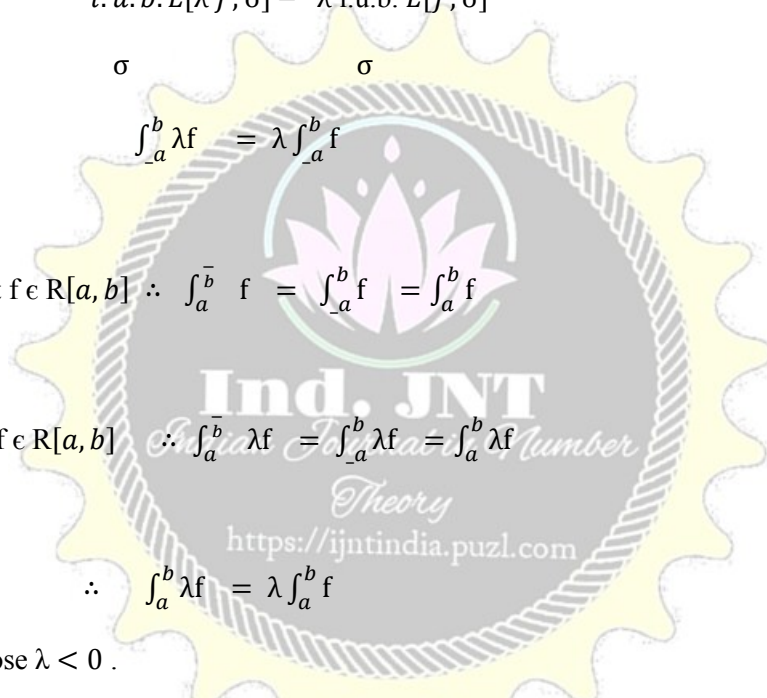
Clearly $g. l. b. \text{ of } \{-f(x)\} = -l. u. b. \{f(x)\}$

$$x \in I_k \qquad x \in I_k$$

$$m[-f; I_k] |I_k| = - M[f; I_k] |I_k|$$

$$\sum_{k=1}^n m[-f; I_k] |I_k| = - \sum_{k=1}^n M[f; I_k] |I_k|$$

$$L[-f; \sigma] = - U[f; \sigma]$$



$$l.u.b.L[-f; \sigma] = l.u.b.\{-U[f; \sigma]\}$$

$$x \in I_k \qquad x \in I_k$$

$$l.u.b.L[-f; \sigma] = \{-g.l.b.U[f; \sigma]\}$$

$$\int_a^b f = - \int_a^{\bar{b}} f$$

Since $f \in R[a, b]$ and $-f \in R[a, b]$

$$\therefore \int_a^{\bar{b}} f = \int_a^b f \text{ and } \int_a^{\bar{b}} -f = \int_a^b -f = \int_a^b -f$$

$$\therefore \int_a^b -f = - \int_a^b f \dots\dots\dots(3)$$

Since $\lambda < 0$ Put $\lambda = -\mu$, then $\mu > 0$

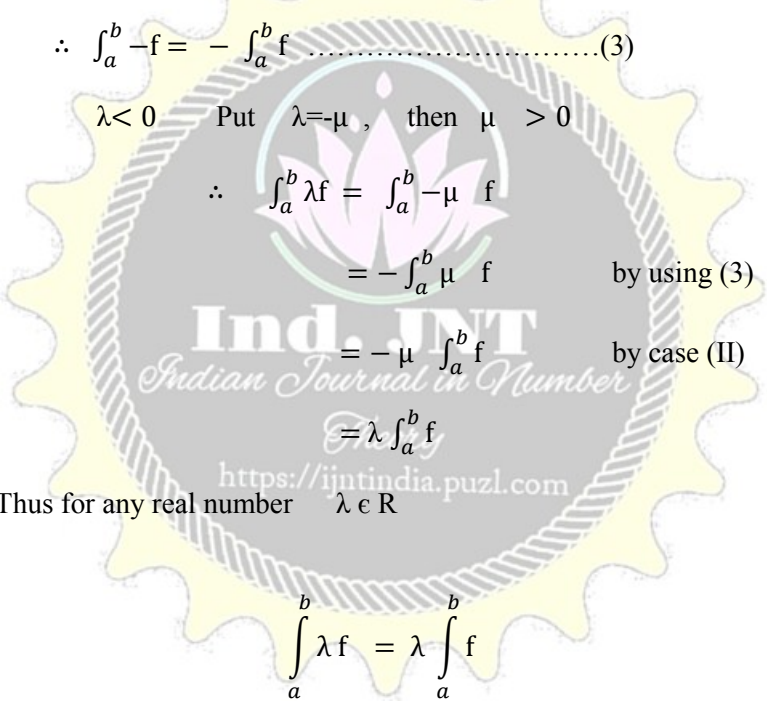
$$\begin{aligned} \therefore \int_a^b \lambda f &= \int_a^b -\mu f \\ &= - \int_a^b \mu f \qquad \text{by using (3)} \end{aligned}$$

$$= -\mu \int_a^b f \qquad \text{by case (II)}$$

$$= \lambda \int_a^b f$$

Thus for any real number $\lambda \in R$

$$\int_a^b \lambda f = \lambda \int_a^b f$$



References:

- [1] D. Somasundaram and B. Chaudhari: A First Course in Mathematical Analysis, Narosa Publishing House (1997)
- [2] R. R. Goldberg: Methods of real Analysis; Oxford & IBH Publishing Co. (1970)

