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ITERATIVELY MINIMUM MEAN SQUARE ERROR (MMSE)-ALGORITHMICALLY MODIFIED MEYER-KONIG AND ZELLER OPERATOR POLYNOMIAL FOR EFFICIENT POLYNOMIAL APPROXIMATION

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Abstract- The celebrated **Weirstrass, K. (1885)'s** theorem heralded an intermittent interest in polynomial approximation in approximation theory. Bernstein polynomial approximation operator was very popular for quite some time. Many modifications were tried by altering the weight-function therein, including some with probabilistic perspective it had. Szasz (1950) proposed the generalization of the well-known Bernstein polynomial approximation operator extending it to an infinite interval. Heinz-Gerd Lehnhoff (1981) proposed the "Modified Szasz-Mirakjan Operator" for bounded continuous functions 'f(x)'; x€ C [0, 1]. Analogously to "Modified Szasz-Mirakjan Operator", we have proposed a finite version of our modification of the well-known 'Meyer-Konig and Zeller (MKZ) Operator Polynomial, say "Modified Meyer-Konig and Zeller (MKZ) Operator Polynomial" for bounded continuous functions 'f(x)'; x€ C [0, 1/2]. Some iterative algorithms have also been tried for various "Positive Linear Operators". In this paper one such algorithm, using the 'Iteratively Minimum Mean Square Error (MMSE) Algorithmic Modified Meyer-Konig and Zeller (MKZ) Operator Polynomial' has been proposed and studied. The paper includes an 'Empirical Simulation Study' to bring forth the extent of 'Relative Gain in Efficiency' in approximation at each iteration, relative to the original and the proposed 'Iterative' 'Polynomial Approximation Algorithm' using 'Modified Meyer-Konig and

Zeller (MKZ) Operator Polynomial' developed & studied for some example-functions " $(10)^x$, sin (1+x), exp (x), & ln(2+x) ". Maple 17 has been used in this simulation study.

Key Words & Phrases: Polynomial Approximation, Positive Linear Operators, Iteratively Minimum Mean-Squared-Error (IMMSE), Empirical Simulation Study.

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INTRODUCTION AND PROPOSITIONS:

Lot of problems in Science & Engineering could, essentially, be formulated as optimizing and approximation {[6], [12], [15], [20], [23], [28], [29] & [31]}. One would good to peruse [4] & [5], in this context. The celebrated **Weirstrass, K. (1885)'s** {[49]} theorem proved that any continuous function could be approximated by a suitable-degree polynomial, as closely as one is pleased with. Polynomial functions are, of course, extremely well-behaved. One of the proofs {[1], [2]} was based on the popular 'Bernstein Polynomial':

B_n (f; x) $=\sum_{k=0}^{k=n} w_k(x).f(k/n)$; w_k (x) $=\left({}^nC_k\right).x^k.(1-x)^{(n-k)}$; k=0 (1) n being the weight-function, WHERE 'f (.)' is a bounded continuous function in C [0, 1], and "f (k/n)", k=0 (1) n are known values of 'f (x)' at "n+1" equidistant 'Knots'. (1.1)

The '**Weirstrass, K. (1885)'s** theorem' & its proof using ''Bernstein Polynomial' was seminal to a lot of active and intermittent interest in 'polynomial approximation' and in 'Bernstein Polynomial', in particular {[3], [8], [11], [20], [21], [26], [30], [31], [32], [34], [36], [37], [33], [38], [41], [42], [43], [46], [47] & [48]}. This paper, also, concerns with the "Bernstein's Polynomial Approximation Operator". Incidentally, weight function 'w_k(x)' in the "Bernstein's Polynomial Approximation Operator", as in (1.1), has a 'Probabilistic' interpretation being the typical 'Binomial Distribution Term'. Thus, 'B_n (f; x)' is nothing but E (f (x)) ~ 'Mathematical Expectation' of 'f (x)' ~ 'x' following the 'Binomial Distribution' ~ "weighted Average of the n +1 known values with respective weights 'w_k(x)'. Without any loss of generality, [a, b] ~ [0, 1] under suitable transformation of the study-variable. We divide [0, 1] into '*n'* equal intervals, using ‘n+1’equi-distant ‘knots’. Let $xi = i/n$ for $i = 0, 1, ..., n$. If the unknown function is called by ‘f(x)’, the ‘Bernstein’s Polynomial Approximation uses the values f (xi)’s [i = 1, 2, …, n] which are assumed to be known.

Szasz (1950) [44] proposed the following generalization of the well-known Bernstein’s polynomials extending it to the infinite interval:

$$S_n(f;n) = [e^{-nx}\sum_{k=0}^{\infty}\{(nx)^k/k!\}f(k/n)\forall f \in C_{[0,\infty)} \tag{1.2}$$

Heinz-Gerd Lehnhoff (1981) [19,22], in particular, proposed the Modified Szasz-Mirakjan Operator:

$$S_n(f,x) = [\sum_{k=0}^{n} T_k f(k/n)]/\ [\exp(nx)]\forall x \in C[0,1]; f \in C[0,1); T_k = (nx)^k/k!\&k = 0,...,n. \qquad (1.3).$$

This was seminal to a lot of work on Modified Szasz-Mirakjan Operator Polynomial {[5], [11], [19], [22], [25], [33] & [39]}.

Some researchers proposed the following “Modified Szasz-Mirakjan Operator Polynomial” {[17], [19] & [33]} for f(x) continuous & bounded function ‘f(x)’:

$$MS_n(f,x) = [\sum_{k=0}^{n} T_k(x) f(k/n)]/\ [T]\forall x \in C[0,1]; f \in C[0,1); T(x) = \sum_{k=0}^{n}\{(nx)^k/k!\}. \tag{1.4}$$

It is worth noting, in the context of this “Modified Szasz-Mirakjan Operator Polynomial”, that the weight-function adds up to ‘1’, i.e., as follows:

$$\begin{aligned}&T_k(x)/T(x) = w_k(x); say, \forall k = 0(1)n. Therefore, apparently, \sum_{k=0}^{n} w_k(x) = 1\ \&\ MS_n(f,x) \equiv \\ &E\{f(x)\} \equiv \text{"Mathematical Expectation" of "} f(x)\text{" w.r.to" probability function "} w_k(x)\text{'s".}\end{aligned} \tag{1.5}$$

The aforesaid “Modified Szasz-Mirakjan Operator Polynomial” could be invoked to approximate any bounded and continuous function ‘f (x)” using its known values “f (k/n); k = 0 (1) n” at ‘n+1’ equidistant “knots” in the interval [0, 1]. This would be without any loss of generality as the approximation would also hold for x ε [a, b], and it holds conversely. Essentially, x ε [0, 1] and x ε [a, b] are identical, for all practical purposes; they are linearly isometric as normed spaces, order isomorphic as lattices, and isomorphic as algebras (rings).

A modification of the classical Meyer-König and Zeller (MKZ) operators [24, 9, 18, and 45] was defined by Cheney and Sharma [7, 9, 11, 27, and 32] as follows:

$$M_n(f;x) = \sum_{k=0}^{\infty} f(\frac{k}{n+k}).m_{n+1,k}(x); f \in C[0,1)\ \&\ x \in [0,1)\ \&\ m_{n,k} = \binom{n+k-1}{k}.x^k.(1-x)^n.$$

(1.6)

Analogously to (1.4) & (1.5), and without any loss of generality, as noted above, we propose “Modified MKZ Operators” as follows:

$$MM_n(f,x) = [\sum_{k=0}^{n} T_k f(k/(k+n))]/T \;\forall x \in C[0,1/2]; f \in C[0,1/2]; T_k = \binom{n+k-1}{k}.x^k \;\&\; k = 0,\ldots,n. \quad (1.7)$$

It is worth noting, in the context of this "Modified Szasz-Mirakjan Operator Polynomial", that the weight-function adds up to '1', i.e., as follows:

$$T_k(x)/T(x) = w_k(x); say, \forall k = 0(1)n. Therefore, apparently, \sum_{k=0}^{n} w_k(x) = 1 \;\&\; MM_n(f,x) \equiv E\{f(x)\} \equiv \text{"Mathematical Expectation" of "} f(x)\text{" w.r.to " probability function " } w_k(x)\text{'s"}. \quad (1.8)$$

Further, we have proposed and studied [via 'Empirical simulation study'], in what follows a 'Computerizable Iterative Algorithmic Minimum Mean Squared Error [MMSE] with the motivation of having an improved approximation by our "**Modified MKZ Operator Polynomial**", at each iteration.

2. The Iterative Algorithmic Minimum Mean Square Error (MMSE) Modified MKZ Operator Polynomial.

Now, we propose our 'Iterative Minimum Mean Square Error (MMSE) Algorithmic **Modified MKZ Operator** Polynomial', using the statistical perspective of 'Mean Square Error (MSE)'. The 'Original **Modified MKZ Operator** Polynomial' in (1.4) is the "MMSE **Modified MKZ Operator** Polynomial at the **Iteration 'Zero'**", i.e. say, **MMSE MM$_n$ [0] (f; x) [≡ MM$_n$ (f; x)]** is an estimate of the **unknown function *f* (*x*),** using its known values "f (k/(k+n)); k = 0 (1) n" at 'n+1' equidistant "knots" in the interval [0, 1/2]'. First, we embark upon the **"Iteration # 1"** of our proposed 'Iterative Algorithm'. The concept "Minimum Mean Square Error Estimator **(MMSEE)**"of Searles (1964) [40] is seminal to this "Iteration # 1".

As per (1.4), "**Modified MKZ Operator** Polynomial" is an estimator analogous to the sample-mean' $\bar{x}$ '. Searles (1964) considered the class of estimators **'$k.\bar{x}$',** and choose the "Optimal" value say "k_0" by minimizing the MSE ($k.\bar{x}$) to be led to the **MMSEE ($k_0.\bar{x}$) ≡ the 'MMSE Estimator'**.

Similarly to Searles [40], we use the perturbed 'Polynomial', say **b. MMSE MM$_n$ [0] (f; *x*),** and hence determine the estimated values of the unknown function '*f* (*x*)' at the knots '(k/(k+n))', say *Et [1] f* (k/(k+n)), *k* = 0 (1) *n*, vis-`a-vis the known values of the unknown function "*f* (*x*)", namely '*f* (k/(k+n))'; *k* = 0 (1) *n*. Hence the **"Knot-Wise Squared-Error", say, *E1r f* (k/(k+n)) ≡ [*Et [1] f* (k/(k+n)) − *f* (k/(k+n))]^2, *k* = 0 (1) *n*** could be generated to lead to the construction of the "Squared-Error Polynomial Function". This will be a "Quadratic Polynomial in *b*", say ***Q* (*b*) ≡ $A.b^2 + B.b + C$.**

To avoid any complex solution to $Q(b) = 0$, we choose $\boldsymbol{b_0 = -(B/2.A)}$ to minimize the value of MSE, **leading to a 'Reduced-MSE Polynomial' estimator "b_0. MMSE MM_n [0] (f*;* *x*)". This completes the "FIRST Iteration".** Consequently, we arrive at the end of our "FIRST Iteration" [after the achievement through the 'Statistical Perspectives of 'MSE'] to propose our **'MMSE Modified MKZ Operator Polynomial'**, **after "FIRST Iteration" as: ~**

MMSE MM_n [1] (f; x) ≡ b_0. MMSE MM_n [0] (f*;* *x*). (2.1)

Next, to arrive at the end of our "SECOND Iteration" [after the achievement through the 'Statistical Perspective of 'MSE'] we ought to use our **'MMSE Modified MKZ Operator'**, after **"FIRST Iteration".**

Likewise we would have, simply, to be putting our 'MMSE **Modified MKZ Operator'**, after the "FIRST Iteration", namely MMSE MS_n [1] (f; x), in the shoes of MMSE MS_n [0] (f; x), and would be proceeding exactly similarly. Apparently, therefore, we could, exactly analogously, proceed to have as many 'Iterations' as we please.

3. The empirical simulation study.

To illustrate the gain in efficiency of the **'MMSE MM_n [1] (f; x)',** and subsequently by using the operator **MMSE MM_n [I] (f; x),** after each 'Iteration" [I ≡ **2,3, …]** of our proposed Iterative Algorithmic Improvement of Polynomial Approximation, we have carried out an empirical study. We have taken the ***cases of n = 2, 3, 4 and 5 (i.e. n + 1 = 3, 4, 5 and 6 knots)*** in the empirical study.

One of the purposes is to numerically illustrate the relative gain in efficiency in using the Algorithm Vis-`a-Vis the **Original 'MMSE MM_n [0] (f; x)' & 'MMSE MM_n [I] (f; x); I ~ Iteration # = 1, 2, ….,** for each example-case of the n-values. Essentially, the empirical study is a simulation one in which inasmuch as we assume that the function to be approximated, namely $f(x)$, is known to us. Once again, we have confined ourselves to illustrating relative gain in efficiency by Iterative Improvement for the following four functions:

$\boldsymbol{f(x) = (10)^x}$**, sin (1+x), exp (x), & ln (2+x).**

To illustrate the potential improvement with our proposed Algorithm, we have considered ***FIVE*** Iterations (for simplicity), and the numerical values of the ***ELEVEN*** **quantities. We have *SIX* quantities ~ FIVE Percentage Relative Errors (*PREs*)** corresponding to Improvement Iteration **(I ≡ 1, *or* 2, *or* 3, *or* 4, *or* 5) ~ *PRE*** **{MMSE MM_n [I] (f; x)} & ONE** corresponding to **the Original 'Modified MKZ Operator' *PRE*** **{MMSE MM_n [0] (f; x)}**.

And we have *FIVE* quantities ~ corresponding Percentage Relative Gains (*PRGs*) in using our Iterative Algorithmic 'MMSE Modified MKZ Operator MMSE MM_n [I] (f; x)}' in place of the Original 'Modified MKZ Operator' MMSE MM_n [0] (f; x), namely *PRG* {MMSE MM_n [I] (f; x)}; I = 1 (1) 5).

These quantities are defined as follows.

The *PRE* using (Original & Iterative) Modified MKZ Operator (Polynomial), namely "{MMSE MM_n [0] (f; x)}" & "{MMSE MM_n [I] (f; x)}" using n intervals in [0, 1], i.e. [(*k* − 1)/(k+*n*), *k*/(*k*+*n*)]; *k* = 1 (1) n*:*

$$PRE\ \{\text{MMSE } MM_n\ [I]\ (f;x)\} = \left\{ \frac{abs[\int_0^{1/2} f(x)dx - \int_0^{1/2} \{\text{MMSE MMn [I](f; x)}\}\, dx]}{\int_0^{1/2} f(x)dx} \right\} \text{ x } 100\%;$$

I ≡ 0 (1) 5. (3.1)

The ***PRG*** by using the Improvement Iteration **(*I* # 1, *or* 2, 3 *or* 4 *or* 5) MMSE MS_n [I] (f; x)** over using the "Original" **Modified MKZ Operator** (Polynomial), using n intervals in [0, 1], i.e.

***PRG* {MMSE MM_n [I] (f; x)}** =

$$= \left[\frac{abs[\text{PRE}\{\text{MMSEMMn[0](f; x)}\} - \text{PRE}\{\text{MMSEMMn[I](f; x)}\}]}{\text{PRE}\{\text{MMSEMMn[0](f; x)}\}} \right] \text{ x } 100\%;$$

I ≡ 1 (1) 5. (3.2)

CONCLUSION:

Thus, **ELEVEN** numerical quantities have been computed using ***Maple Release 17***, for all the **four** illustrative functions (**$(10)^x$, sin (1+x), exp (x), & ln (2+x)**) mentioned in Section 3, and for **four** values of *n* (***n* = *2, 3, 4, and 5***) These values are, respectively, tabulated in **Tables A.1– A.4 [Appendix].**

The **PREs** for our **'Iteratively Algorithmic MMSE Modified MKZ Operator Polynomial'** Approximators are PROGRESSIVELY lower on each subsequent iteration, as compared to that for the Original **Modified MKZ Operator** Polynomial Approximator, for all the illustrative functions. The **PRGs** due to the use of our proposed **'Iteratively Algorithmic MMSE Modified MKZ Operator Polynomial'** Approximators rather than that of the Original **Modified MKZ Operator** Polynomial Approximator are also PROGRESSIVELY increasing on each subsequent iteration, for all the illustrative functions.

Lastly, it is very heartening to note that ***when we use (n = 5) intervals, i.e. SIX KNOTS*** for the polynomial approximation, the *PRG* becomes ***quite close to (99%)*** **{Table A.1; f (x) = 10^x}; to *(96%)* {Table A.2; f (x) = sin (1 +x)} to around (*92%)* {Table A.3; f (x) = exp (x)} & around *(98%)* {Table A.4; f (x) = ln (2+x)} even with n =2, i.e. only for THREE 'Knots',** for the **fifth** iteration.

Otherwise also, the speed of convergence is highly accelerated by the Iterative Algorithmic improvement by proposed MMSE **Modified MKZ Operator** Polynomial using the Statistical perspective of **"MMSE"**.

In fact, the improvement is conspicuously missing when we use more "Knots" indicating that our proposed "**Modified MKZ Operator" lacs in using the additional information [KNOWN VALUES OF THE UNKNOWN FUNCTION 'f (X)', AT THE ADDITIONAL KNOTS], and rather uses it negatively in terms of 'efficiency of approximation'. This is an "Open Problem" to suitably improve our prosed "Modified MKZ Operator", and we are working on it.**

It could also be noted that this perspective of the Iterative Improvement could be applied to any Polynomial Approximator, other than **Modified MKZ Operator** Polynomial; more particularly to those belonging to the class of Positive Linear Operators, as they admit to **the Statistical perspective 'MMSE'** rather more readily!

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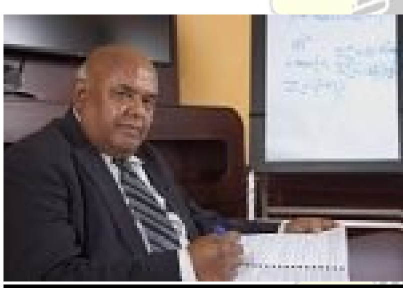

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APPENDIX

Table A.1. MMSE[k] Modified* MKZ *Polynomial REs & RGs (%) for f (x) =10^x^.

Items↓ n →	2	3	4	5
PRE {MMSE[0] MM_n (f; x)}	17.25201865	26.73607872	31.63633776	34.74714387
PRE {MMSE[1] MM_n (f; x)}	5.698524532	9.327458440	16.77087213	22.28290853
PRE {MMSE[2] MM_n (f; x)}	5.186515503	1.823502370	6.937391099	11.99201857
PRE {MMSE[3] MM_n (f; x)}	4.286710656	0.293715041	2.440802846	5.373652818
PRE {MMSE[4] MM_n (f; x)}	3.421951583	0.234480201	0.794643722	1.918501550
PRE {MMSE[5] MM_n (f; x)}	2.409317473	0.023735550	0.245919097	0.367882741
PRG {MMSE[1] MM_n (f; x)}	66.96894059	65.11284045	46.98857922	35.87125142
PRG {MMSE[2] MM_n (f; x)}	69.93676157	93.17961924	78.07144698	65.48775745
PRG {MMSE[3] MM_n (f; x)}	75.15241116	98.90142813	92.28481227	84.53497980
PRG {MMSE[4] MM_n (f; x)}	80.16492070	99.12298209	97.48819308	94.47867843
PRG {MMSE[5] MM_n (f; x)}	86.03457648	99.91122277	99.22266888	98.94125761

Table A.2. MMSE[k] Modified* MKZ *Polynomial REs & RGs (%) for f (x) = sin (1+x).

Items↓ n →	2	3	4	5
PRE {MMSE[0] MM_n (f; x)}	4.840209014	6.037925614	6.749330224	7.258985154
PRE {MMSE[1] MM_n (f; x)}	2.130142681	3.353415907	4.173186684	4.841599599
PRE {MMSE[2] MM_n (f; x)}	0.977538420	1.836923703	2.458645520	3.055250288
PRE {MMSE[3] MM_n (f; x)}	0.509016509	1.020908082	1.367368325	1.781382757
PRE {MMSE[4] MM_n (f; x)}	0.313713516	0.590890928	0.696464936	0.901601921

PRE {MMSE[5] MM_n (f; x)}	0.223984830	0.364459536	0.296331433	0.312757376
PRG {MMSE[1] MM_n (f; x)}	55.99068811	44.46079463	38.16887683	33.30197684
PRG {MMSE[2] MM_n (f; x)}	79.80379737	69.57690736	63.57200732	57.91077922
PRG {MMSE[3] MM_n (f; x)}	89.48358413	83.09174132	79.74068123	75.45961702
PRG {MMSE[4] MM_n (f; x)}	93.51859568	90.21367659	89.68097703	87.57950455
PRG {MMSE[5] MM_n (f; x)}	95.37241410	93.96382865	95.60946906	95.69144489

Table A.3. MMSE[k] Modified* MKZ *Polynomial REs & RGs (%) for f (x) = exp(x).

Items↓ *n* →	2	3	4	5
PRE {MMSE[0] MM_n (f; x)}	9.128373059	13.15674894	15.31341372	16.72576296
PRE {MMSE[1] MM_n (f; x)}	1.696043046	6.479021977	9.379989288	11.51086949
PRE {MMSE[2] MM_n (f; x)}	1.393410546	2.872996484	5.340016285	7.470153637
PRE {MMSE[3] MM_n (f; x)}	1.388068837	1.212402530	2.869432180	4.560482512
PRE {MMSE[4] MM_n (f; x)}	1.235769715	0.513132828	1.467266610	2.589618263
PRE {MMSE[5] MM_n (f; x)}	0.822684432	0.233891637	0.709417112	1.320128657
PRG {MMSE[1] MM_n (f; x)}	81.42009496	50.75514471	38.74658218	31.17880770
PRG {MMSE[2] MM_n (f; x)}	84.73539001	78.16332520	65.12850510	55.33744168
PRG {MMSE[3] MM_n (f; x)}	84.79390765	90.78493832	81.26196920	72.73378487
PRG {MMSE[4] MM_n (f; x)}	86.46232240	96.09985086	90.41842246	84.51718904
PRG {MMSE[5] MM_n (f; x)}	90.98761163	98.22226873	95.36734837	92.10721410

Table A.4. MMSE[k] Modified* MKZ *Polynomial REs & RGs (%) for f (x) = ln (2+x).

Items↓ n →	2	3	4	5
PRE {MMSE[0] MM_n (f; x)}	6.115014575	8.253251183	9.439385822	10.24208023
PRE {MMSE[1] MM_n (f; x)}	2.017429072	4.409979783	5.894922404	7.022529150
PRE {MMSE[2] MM_n (f; x)}	0.405972054	2.265549087	3.510011115	4.599384506
PRE {MMSE[3] MM_n (f; x)}	0.312666761	1.160760093	1.998472607	2.853902220
PRE {MMSE[4] MM_n (f; x)}	0.292903659	0.614710611	1.081046068	1.642598460
PRE {MMSE[5] MM_n (f; x)}	0.139114845	0.349245575	0.542038528	0.829342785
PRG {MMSE[1] MM_n (f; x)}	67.00859749	46.56675672	37.54972500	31.43454267
PRG {MMSE[2] MM_n (f; x)}	93.36106155	72.54961667	62.81525958	55.09325837
PRG {MMSE[3] MM_n (f; x)}	94.88690081	85.93572318	78.82836188	72.13552173
PRG {MMSE[4] MM_n (f; x)}	95.21009058	92.55189746	88.54749569	83.96225744
PRG {MMSE[5] MM_n (f; x)}	97.72502840	95.76838821	94.25769284	91.90259433