Ind. JNT
(Indian Journal in Number theory)

Received: *13.10.2015* ***Year:*** *2017*, ***Pages:*** *99-102*
Published: *30.11.2017* *Original Article***

PROOF OF ANOTHER COUNTING PRINCIPLE

A. D. Chindhe[1] <anilchindhe5@gmail.com>

Department of Mathematics,

Balbhim College, Beed.

Abstract- In the books of Abstract algebra 'Another counting Principle for order of subgroups' is proved using Centralizer of Group. In this paper I proved 'Another counting Principle' using Lagrange's theorem directly. If H and K are finite subgroups of G of orders O (H) and O (K) respectively, then O (HK) is given by $\frac{O(H).O(K)}{O(H\cap K)}$ this result has special importance in Group theory.

Key words: Group, Subgroup, Coset

INTRODUCTION:

Definition: If H, K are two subgroups of G, let $HK = \{x \in G | x = hk, h \in H, k \in K\}$.

Lemma: HK is a subgroup of G iff HK = KH.

Lemma: Let G be a group & H be a subgroup of G. Then G is a union of all left (right) cosets of H in G.

Lemma: Any two left (right) cosets of H in G are disjoint.

Lemma: Any two left (right) cosets of H in G have the same (finite or infinite) number of elements.

Lagrange's theorem: 'Let G is a finite group & H is a subgroup of G, then the order of H divides the

Order of G'

MAIN RESULT

Theorem : If H and K are finite subgroups of G of orders O(H) and O(K) respectively,

then $O(HK) = \frac{O(H).O(K)}{O(H \cap K)}$

Proof:

Let $I = H \cap K$. I is a subgroup of G and, since $I \subseteq K$, I is a subgroup of K. Let $IK_1, IK_2, IK_3, \ldots\ldots, IK_n$ be the n distinct cosets of I in K. Thus

$$K = IK_1 \cup IK_2 \cup IK_3 \cup \ldots \cup IK_n$$

$$|K| = |IK_1| + |IK_2| + |IK_3| + \ldots + |IK_n|$$

$$|K| = n|I|$$

$$n = \frac{|K|}{|I|} = \frac{|K|}{|H \cap K|}$$

We now claim that $HK = HK_1 \cup HK_2 \cup HK_3 \cup \ldots \cup HK_n$.

For if $hk \in HK$, then $k = lk$, for some $l \in I$, j is an integer between 1 and n.

Hence $hk = (hl)\, kj = h^l k_j$ where $h^l \in H$, as both h, l belong to H.

Thus $HK = HK_1 \cup HK_2 \cup HK_3 \cup \ldots \cup HK_n$

Now suppose $Hki \cap Hk_j \neq \emptyset$ for some integers i and j.The $hk_i = h^l k_j$ for some $h, h^l \in H$.

Consequently $(h^i)^{-1}h = kj\,(k_i)^{-1}$, so $kj\,(k_i)^{-1} \in I = H \cap K$.But $kj\,(k_i)^{-1} \in I$ implies that $k_j \in IK_i$.

Since two cosets are either equal or disjoint, $IKj = IK_i$. Hence $k_i = k_j$.

Thus $Hki \cap Hk_j = \emptyset$ for $i \neq j$ and

$$|HK| = |HK_1| + |HK_2| + |HK_3| + \ldots + |HK_n|$$

Now $|HK| = |HK_i|$, because $h_1k_i = h_2k_i$ if and only if $h_1 = h_2$

Therefore

$$|HK| = n|H| = \frac{|H|\,|K|}{|H \cap K|}, \text{ since } n = \frac{|K|}{|H \cap K|}$$

Hence $O(HK) = \frac{O(H).O(K)}{O(H \cap K)}$

Corollary:

Let G be a finite group with normal subgroups H and K where $|H|\,|K| = |G|$. If either $H \cap K = (e)$ or $HK = G$, then $G \cong H \times K$

References:

[1]. **Topics in Algebra**, I. N. Herstein, John Wiley and Sons (Second Edition) 2006.

[2]. **Algebra Vol-1-Groups** Vivek Sahai & Vikas Bist Narosa Publicationing house New Delhi1996.

[3]. **Abstract Algebra**, John B. Fraleigh, Pearson Edition 2009.